Week 1: Finite Automata

Proofs by induction

In these exercices, $\mathbb{N}$ is the set of all integers $\{0, 1, 2, \dots\}$ (see page 22 of the text book: integers as recursively defined concepts)

1. Define $f : \mathbb{N} \to \mathbb{N}$ by recursion

   $$f(0) = 0 \qquad f(n+1) = f(n) + n$$

   What is $f(2)$? $f(3)$? Use mathematical induction to show that for all $n \in \mathbb{N}$ we have

   $$2f(n) = n^2 - n$$

2. Suppose that we have stamps of 4 kr and 3 kr. Show that any amount of postage over 5 kr can be made with some combinations of these stamps.

3. We define by recursion

   $$0! = 1 \qquad (n+1)! = (n+1) \times n!$$

   Show that $n! \geq 2^n$ for $n \geq 4$ by analogy with the proof of example 1.17, page 21 of the text book.

4. Define the two functions $f, g : \mathbb{N} \to \mathbb{N}$ by recursion

   $$f(0) = 0 \quad g(0) = 1 \qquad f(n+1) = g(n) \qquad g(n+1) = f(n)$$

   What is $g(2)$? $f(4)$? Show by mathematical induction that we have, for all $n \in \mathbb{N}$

   $$f(n) + g(n) = 1 \qquad f(n)g(n) = 0$$

   Show that $f(n) = 0$ iff $g(n) = 1$ iff n is even and $f(n) = 1$ iff $g(n) = 0$ iff n is odd by *mutual* induction, by analogy with the proof page 26-27-28 in the text book.

5. We define

   $$f(0) = 0,\ f(1) = 1,\ f(n+2) = f(n+1) + f(n)$$

   (Fibonacci function). We define then $s(0) = 0,\ s(n+1) = s(n) + f(n+1)$. Prove by induction that we have

   $$\forall n. s(n) = f(n+2) - 1.$$

We define then

$$l(0) = 2, \; l(1) = 1, \; l(n+2) = l(n+1) + l(n)$$

Prove by induction that we have $l(n+1) = f(n) + f(n+2)$.

6. We define a_2 and a_1 given x_1 and x_2

$$a_1(0) = 0 \qquad a_1(t+1) = \neg(a_2(t) \vee x_1(t))$$

$$a_2(0) = 1 \qquad a_2(t+1) = \neg(a_1(t) \vee x_2(t))$$

We may think of this as a digital circuit. The inputs are $x_1(t)$ and $x_2(t)$ and the output is $a_2(t)$. The predicate $a_1(t)$ represents an internal state of the circuit. Let 0 mean that the voltage is *low* and 1 mean that the voltage is *high*. The voltage at the input may take any value at any time. Compute the values of $a_1(t)$ and $a_2(t)$ for the following sequences

$$x_1 = 000000000111000\ldots \qquad\qquad x_2 = 00111000000000\ldots$$

that is $x_1(0) = x_1(1) = \cdots = x_1(6) = 0, \; x_1(7) = x_1(8) = x_1(9) = 1, \ldots$ If we call a *pulse* of high voltage a sequence of three 1s, explain why this circuit can be called a *memory* (it *remembers* which input pulsed last).

7. If $\Sigma = \{a, b, c\}$ what is Σ^1? Σ^2? Σ^0?

8. If $\Sigma = \{0, 1\}$ find a counterexample to the following alleged theorem: for all $x, y \in \Sigma^*$ we have

$$x^2 y = xyx$$

(cf. section 1.3.4)

9. Let $\Sigma = \{0, 1\}$ we define $\phi : \Sigma^* \to \Sigma^*$ by recursion

$$\phi(\epsilon) = \epsilon \qquad \phi(w0) = \phi(w)1 \qquad \phi(w1) = \phi(w)0$$

What is $\phi(1011), \; \phi(1101)$? Show that

$$|\phi(w)| = |w|$$

by induction on $|w|$.

10. Let $\Sigma = \{0, 1\}$ we define the reverse function on Σ^* by the laws

$$rev(\epsilon) = \epsilon \qquad rev(ax) = rev(x)a$$

What is $rev(010)$? $rev(10)$?

Show by induction on y that we have

$$rev(yx) \ = rev(x)rev(y)$$

Show by induction on $n \in \mathbb{N}$ that we have

$$rev(x^n) = (rev(x))^n$$

11. When can we have $x^2 = y^3$ with $x, y \in \Sigma^*$, where Σ is a finite set?