Prüfer domain

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Valuation domain

A valuation domain is an integral domain R such that for any u, v in R either v divides u or u divides v

Another formulation is that for any $s \neq 0$ in the field of fraction of R we have s in R or $1/s$ in R

Theorem: *A valuation domain is integrally closed*

Indeed assume $s \neq 0$ is integral over R. We have an equation

$$s^n + a_1 s^{n-1} + \cdots + a_0 = 0$$

Then either s is in R (and we have finished) or $1/s$ is in R. But we have $s = a_1 + a_2/s + \cdots + a_0/s^{n-1}$ and hence s is in R

Prüfer domain

Classicaly a Prüfer domain R is a domain R such that for any prime $\mathfrak{p}$ of R the localisation $R_\mathfrak{p}$ is a valuation domain

This means that for any $u, v \neq 0$ in R then we have v/u in $R_\mathfrak{p}$ or u/v in $R_\mathfrak{p}$

How to write this in a finite way (without points)?

We remark that if we have v/u in $R_\mathfrak{p}$ then there exists a in R such that $\mathfrak{p}$ is in $D(a)$ and v/u is in $R[1/a]$

Prüfer domain

Hence for any u, v and any $\mathfrak{p}$ there exists a such that $\mathfrak{p}$ is in $D(a)$ and v/u is in $R[1/a]$ or u/v is in $R[1/a]$.

By compactness of the Zariski spectrum we have finitely many elements $a_1, \dots, a_n$ in R such that $1 = D(a_1, \dots, a_n)$ and for each i, we have u/v is in $R[1/a_i]$ or v/u is in $R[1/a_i]$.

This is a finite condition but we can simplify it a little

We can first assume $\Sigma a_i = 1$. Then taking b to be the sum of all a_i such that u/v is in $R[1/a_i]$ we see that u/v is in $R[1/b]$ and v/u is in $R[1/1-b]$

We have used the fact that if $u_1/v_1 = u_2/v_2$ then $u_1/v_1 = u_2/v_2 = u_1 + u_2/v_1 + v_2$

Prüfer domain

Thus we get the point-free condition: for any u, v we can find b such that u/v is in $R[1/b]$ and v/u is in $R[1/1-b]$

This means $u/v = p/b^N$ and $v/u = q/(1-b)^N$ for some N

Since $1 = D(b^N, (1-b)^N)$ we can still simplify this to $u/v = d/c$ and $v/u = e/1-c$

This gives the other equivalent condition: for any u, v there exists c, d, e such that $uc = vd$ and $v(1-c) = eu$

Notice that this is a simple first-order (and even coherent) condition

A ring satisfying this condition is called *arithmetical*

Local-global principal

Let R be a Prüfer domain

We know that, locally, R is a valuation domain

We know also that a valuation domain is integrally closed

Hence we deduce from a local-global principle that R is integrally closed

We can follow this reasoning and get a direct proof that R is integrally closed from the fact that R is arithmetic (this is yet another illustration of the completeness of coherent logic)

Dedekind Domain

Classically a Dedekind Domain can be defined to be a *Noetherian* Prüfer domain

A Noetherian valuation domain is exactly a *discrete* valuation domain, which happens to be of Krull dimension $\leqslant 1$

Hence (local-global property) a Dedekind domain is of Krull dimension $\leqslant 1$: a non zero prime ideal is maximal

But several important properties of Dedekind domain hold already for Prüfer domain, which is a *first-order notion* (and which is not necessarily of dimension $\leqslant 1$)

Principal Localization Matrix

A valuation domain is such that the divibility relation is linear

Hence if we have finitely many element $x_1, \ldots, x_n$ one of them divides all the other

Over a Prüfer domain R we deduce that we have $a_1, \ldots, a_n$ such that $1 = D(a_1, \ldots, a_n)$ and x_i divides all x_j in $R[1/a_i]$

As before we can simplify this condition by $1 = \Sigma a_i$ and there exists b_{ij} such that $b_{ij}x_j = a_i x_i$

Principal Localization Matrix

In this way we get the existence of a matrix a_{ij} such that $1 = \Sigma a_{ii}$ and $a_{ij}x_j = a_{ii}x_i$

Such a matrix is called a *principal localization matrix* of the sequence $x_1, \ldots, x_n$

If all x_i are $\neq 0$ we get $a_{ji}x_j = a_{jk}x_i$ and we have

$$<a_{1i}, \ldots, a_{ni}><x_1, \ldots, x_n> = <x_i>$$

In particular we have an *inverse* of the ideal $<x_1, \ldots, x_n>$ (the product is a non zero principal ideal)

Inverse of finitely generated ideal

Dedekind himself thought that the existence of such an inverse was *the* fundamental result about the ring of integers of an algebraic field of numbers (see J. Avigad's historical paper on Dedekind)

Our argument is constructive, thus can be seen as an *algorithm* which computes this inverse over an arbitrary Prüfer domain

All we need is to know constructively

$$\forall x\ y. \exists\ u\ v\ w. \quad xu = yv \wedge y(1-u) = xw$$

Application

If $I \subseteq J$ are 2 f.g. ideals we can compute a f.g. ideal K such that $J.K = I$

Indeed this is simple if J is principal, and we can find J' such that $J.J'$ is principal, and then $I.J' \subseteq J.J'$

In particular, if I, J are f.g. ideals since we have $I.J \subseteq I + J$ we can find K f.g. such that $I.J = (I + J).K$. It follows then that $K = I \cap J$

Hence: the intersection of two f.g. ideals is f.g. and we have an algorithm to find the generators of this intersection

Application

This can be stated as: *any Prüfer Domain is coherent*

Classically one works with Dedekind Domain, that are Noetherian, and this remarkable fact is usually not stressed (Noetherian implies coherent in a trivial way)

The center map for a Prüfer Domain

Theorem: *If R is a Prüfer Domain then the center map ψ : Zar$(R) \rightarrow$ Val(R) is an* isomorphism

We show that ψ is surjective

We consider $s = x/y$ with x, y in R

We have u, v, w such that $ux = vy$ and $(1-u)y = wx$

We can then check that we have $V_R(x/y) = \psi(D(u, w))$ and $V_R(y/x) = \psi(D(1-u, v))$

The center map for a Prüfer Domain

It may be that ψ is surjective but R is not a Prüfer Domain

An example is $R = \mathbb{Q}[x, y]$ with $y^2 = x^3$ which is not integrally closed

Proposition: *If R is integrally closed and the center map is surjective then R is a Prüfer Domain*

Gilmer-Hoffmann's Theorem

We present now a simple sufficient condition for R to be a Prüfer domain

For any non zero s in the field of fraction of R we have to find u, v, w in R such that $u = vs$ and $(1-u)s = w$

Theorem: *If s is a zero of a primitive polynomial in $R[X]$ then we can find* u, v, w integral over R *such that* $u = vs$ *and* $(1-u)s = w$

This is a fundamental result for producing integral elements

Gilmer-Hoffmann's Theorem

We write $a_n s^n + \cdots + a_0 = 0$ with $a_n, \ldots, a_0$ in R such that $1 = D(a_n, \ldots, a_0)$

We define

$$b_n = a_n,\ b_{n-1} = b_n s + a_{n-1},\ \ \ldots,\ b_1 = b_2 s + a_1$$

We then check that $b_n, b_n s, \ldots, b_1, b_1 s$ are all integral over R

We consider the ring $S = R[b_n, b_n s, \ldots, b_1, b_1 s]$. In this ring we have $1 = D(b_n, b_n s, \ldots, b_1, b_1 s)$ and we have s in $S[1/b_i]$ and $1/s$ in $S[1/b_i s]$

Hence we can find u, v, w in S such that $u = vs$ and $(1-u)s = w$

Applications

Theorem: *If S is the integral closure of a Bezout Domain R in a field extension of its field of fractions then S is a Prüfer Domain*

Indeed if s is in the field of fractions of S then s satisfies a polynomial equation $a_n s^n + \cdots + a_0 = 0$ with $a_n, \ldots, a_0$ in R such that $1 = D(a_n, \ldots, a_0)$, since R is a Bezout Domain

Two particular important cases are $R = \mathbb{Z}$ (algebraic integers) and $R = k[X]$ (algebraic curves)

Applications

Proposition: *If R is a Prüfer Domain and s is in the field of fraction of R then there exists u, w in R such that $R[s] = R[1/u] \cap R[1/w]$. In particular $R[s]$ is integrally closed, and hence, by the Gilmer-Hoffmann's Theorem, $R[s]$ is a Prüfer Domain*

Indeed the equality $R[s] = R[1/u] \cap R[1/w]$ follows from $us = v,\ 1 - u = ws$

Algebraic curves

We apply our results to the case of *algebraic curves*: we consider an algebraic extension L of a field of rational functions $k(x)$

If a is an element of L we have an algebraic relation $P(a, x) = 0$.

If x does not appear in this relation then a is algebraic over k: it is a *constant* of L. We let k_0 be the field of constants of L.

If x appears, then x is algebraic over $k(a)$ and a is a *parameter* and then L is algebraic over $k(a)$. We write $E(x_1, \ldots, x_n)$ the elements integral over $k[x_1, \ldots, x_n]$

Algebraic curves

We consider the formal space $X = \mathsf{Val}(L, k)$

Over X we define a sheaf of rings: if U is a non zero element of $\mathsf{Val}(L, k)$ it is a disjunction of elements of the form $V(a_1) \wedge \cdots \wedge V(a_n)$.

We define $\mathcal{O}_X(U)$ to be the set of elements f in L such that $U \leqslant V(f)$ in $\mathsf{Val}(L, k)$

Algebraic curves

Intuitively any f in L is a meromorphic function on the abstract Riemann surface X and $U \leqslant V(f)$ means that f is holomorphic over the open U

In particular we have $\Gamma(X, \mathcal{O}_X) = k_0$

This is an algebraic counterpart of the fact that the global holomorphic functions on a Riemann surface are the constant functions

Algebraic curves

If p is a parameter and b is in $E(p)$ then we have $E(p, 1/b) = E(p)[1/b]$

More generally

$$\Gamma(V(p) \wedge V(1/b_1, \ldots, 1/b_m)) = E(p)[1/b_1] \wedge \cdots \wedge E(p)[1/b_m]$$

Algebraic curves

Since $E(p)$ is the integral closure of the Bezout Domain $k[p]$ we have that $E(p)$ is a Prüfer Domain

Hence the sublattice $\downarrow V(p)$ of $\mathsf{Val}(L,k)$ is isomorphic, via the center map, to $\mathsf{Zar}(E(p))$

The sheaf $\mathcal{O}_X$ restricted to the basic open $V(p)$ is isomorphic to the affine scheme $\mathsf{Zar}(E(p)),\mathcal{O}$

Algebraic curves as schemes

The pair $X, \mathcal{O}_X$ is thus a most natural example of a *scheme*, which is the glueing of two affine schemes

For any parameter p the space X is the union of the two basic open $U_0 = V(p)$ and $U_1 = V(1/p)$

U_0 is isomorphic to Zar$(E(p))$

U_1 is isomorphic to Zar$(E(1/p))$

The sheaf $\mathcal{O}_X$ restricts to the structure sheaf over each open U_i

The Genus of an Algebraic Curve

Following the usual cohomological argument, one can show

Theorem: *The* k_0*-vector space* $H^1(p) = E(p, 1/p)/(E(p) + E(1/p))$ *is* independent *of the parameter* p *and hence defines an* invariant $H^1(X, \mathcal{O}_X)$ *of the extension* L/k

In particular for L defined by $y^2 = 1 - x^4$ we find $H^1(x) = \mathbb{Q}$

For $L = \mathbb{Q}(t)$ we find $H^1(t) = 0$