Addition formula for algebraic functions

January 10, 2018

A letter from Abel to Crelle

Both Abel and Galois based their theory of algebraic functions to the study of sums $\Sigma\varphi(x_i)$ where the summation is extended to the common zeros of the given algebraic curves $f(x,y)=0$ with another curve $\theta(x,y)=0$ with indeterminate coefficients. It is really surprising that this approach, which seemed fundamental, is not followed anymore in the theory of algebraic functions. In this note, I try to explain a special case that Abel [1] described as being one "of the most remarkable case".

We take $P(x)$ a unitary polynomial of degree 6 and we consider the curve

$$f(x,y)=y^2-P(x)=0$$

We consider the other "varying" curve $\theta(x,y)=y-c-c_1x-c_2x^2-x^3=0$, where c,c_1,c_2 are indeterminates, and the system

$$f(x,y)=\theta(x,y)=0$$

It is direct to eliminate y in this system, getting the equation

$$Q(x)=(c+c_1x+c_2x^2+x^3)^2-P(x)=0$$

Since P is unitary, we get a polynomial Q of degree 5. We then consider the decomposition algebra of Q and Abel writes x_1,x_2,x_3,z_1,z_2 the root of the polynomial Q. We write

$$y_1=\psi(x_1),\ y_2=\psi(x_2),\ y_3=\psi(x_3),\ y_4=\psi(z_1),\ y_5=\psi(z_2)$$

where $\psi(x)$ is the polynomial $c+c_1x+c_2x^2+x^3$. The claim is then that

$$(*)\qquad \frac{(\alpha+\beta x_1)}{y_1}dx_1+\frac{(\alpha+\beta x_2)}{y_2}dx_2+\frac{(\alpha+\beta x_3)}{y_3}dx_3+\frac{(\alpha+\beta z_1)}{y_4}dz_1+\frac{(\alpha+\beta z_2)}{y_5}dz_2=0$$

With the notation of Abel, if $\varphi(x)$ denotes the function

$$\int\frac{(\alpha+\beta x).dx}{\sqrt{P(x)}}$$

we have

$$(1)\qquad \varphi(x_1)+\varphi(x_2)+\varphi(x_3)=C-(\varphi(z_1)+\varphi(z_2))$$

Notice that since we have

$$\begin{aligned}Q(x)&=(c+c_1x+c_2x^2+x^3)^2-P(x)\\&=(2c_2-a_5)x^5+(c_2^2+2c_1-a_4)x^4+(2c+2c_1c_2-a_3)x^3\\&\quad+(c_1^2+2cc_2-a_2)x^2+(2cc_1-a_1)x+c^2-a\end{aligned}$$

we know that z_1, z_2 are roots of the following polynomial (there seems to be a sign mistake in Abel's letter)

$$y^2 - (\frac{c_2^2 + 2c_1 - a_4}{a_5 - 2c_2} - x_1 - x_2 - x_3)y + \frac{c^2 - a}{x_1 x_2 x_3 (a_5 - 2c_2)} = 0$$

Intuitively, if we fix x_1, x_2, x_3 we can find z_1, z_2 such that the equation (1) holds. Abel says then that "toute la theéorie de la fonction φ est comprise dans la l'équation (1), car la propriété exprimée par cette équation détermine, comme on peut le démontrer, cette fonction complètement."

What is left to show is the equality $(*)$. This equality can be written

$$\Sigma \frac{(\alpha + \beta x)}{\psi(x)} dx = 0$$

where x ranges over the roots of the poylynomial Q. We have

$$dx = \frac{\partial x}{\partial c} dc + \frac{\partial x}{\partial c_1} dc_1 + \frac{\partial x}{\partial c_2} dc_2$$

and from $Q(x) = 0$ we derive

$$Q'(x) \frac{\partial x}{\partial c} + \frac{\partial Q(x)}{\partial c} = 0$$

and hence

$$Q'(x) \frac{\partial x}{\partial c} + 2(x^3 + c_2 x^2 + c_1 x + c) = 0$$

that is

$$Q'(x) \frac{\partial x}{\partial c} + 2\psi(x) = 0$$

It follows that

$$\Sigma \frac{(\alpha + \beta x)}{\psi(x)} \frac{\partial x}{\partial c} = -2\Sigma \frac{(\alpha + \beta x)}{Q'(x)}$$

and this is 0 (see the next sextion).

Similarly

$$Q'(x) \frac{\partial x}{\partial c_1} + 2x\psi(x) = 0$$

and hence

$$\Sigma \frac{(\alpha + \beta x)}{\psi(x)} \frac{\partial x}{\partial c_1} = -2x\Sigma \frac{(\alpha + \beta x)}{Q'(x)} = 0$$

and finally

$$Q'(x) \frac{\partial x}{\partial c_2} + 2x^2\psi(x) = 0$$

and

$$\Sigma \frac{(\alpha + \beta x)}{\psi(x)} \frac{\partial x}{\partial c_2} = -2x^2\Sigma \frac{(\alpha + \beta x)}{Q'(x)} = 0$$

A usefull Lemma

What we have used is the following Lemma, proved also by Abel. Assume that we have $Q(x) = (x - x_1)\dots(x - x_n)$ and $Q'(x_i) \neq 0$. Let $xF(x) = QR + S$ with $d(S) < n$. We then have

$$\Sigma \frac{F(x_i)}{Q'(x_i)} = R(0)$$

We prove it for indeterminate $x_1, \dots, x_n$ using

$$\frac{xF(x)}{Q(x)} = R(x) + \Sigma \frac{x_i F(x_i)}{(x - x_i)Q'(x_i)}$$

This follows from the interpolation formula, for $d(S) < n$

$$S = \Sigma \frac{S(x_i)Q(x)}{(x - x_i)Q'(x_i)}$$

The simpler case of an elliptic curve

We consider the curve

$$f(x, y) = y^2 - x + x^3 = 0$$

We consider the other "varying" curve $\theta(x, y) = y - c - c_1 x = 0$, where c, c_1 are indeterminates, and the system

$$f(x, y) = \theta(x, y) = 0$$

It is direct to eliminate y in this system, getting the equation

$$Q(x) = (c + c_1 x)^2 - x + x^3 = 0$$

We then consider the decomposition algebra of Q and write x_1, x_2, z the root of the polynomial Q. We write

$$y_1 = \psi(x_1),\ y_2 = \psi(x_2),\ y_3 = \psi(z)$$

where $\psi(x)$ is the polynomial $c + c_1 x$. The claim is then that

$$(*) \qquad \frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dz}{y_3} = 0$$

If $\varphi(x)$ denotes the function

$$\int \frac{dx}{\sqrt{x - x^3}}$$

we have

$$(1) \qquad \varphi(x_1) + \varphi(x_2) + \varphi(z) = C$$

Since we have

$$\begin{aligned} Q(x) &= (c + c_1 x)^2 - x + x^3 \\ &= x^3 + c_1^2 x^2 + (2cc_1 - 1)x + c^2 \end{aligned}$$

we know that

$$z = -\frac{c^2}{x_1 x_2}$$

Intuitively, if we fix x_1, x_2 we can find z such that the equation (1) holds. Also, c and c_1 are determined by the equations

$$y_1 = c + c_1 x_1 \qquad y_2 = c + c_1 x_2$$

so that

$$c_1 = \frac{y_1 - y_2}{x_1 - x_2} \qquad c = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

We can rewrite

$$c = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} = \frac{x_1 x_2 (1 + x_1 x_2)}{x_1 y_2 + x_2 y_1}$$

so that

$$z = -\frac{c^2}{x_1 x_2} = \frac{x_1 x_2 (1 + x_1 x_2)^2}{(x_1 y_2 + x_2 y_1)^2}$$

What is left to show is the equality $(*)$. This equality can be written

$$\Sigma \frac{dx}{\psi(x)} = 0$$

where x ranges over the roots of the poylynomial Q. We have

$$dx = \frac{\partial x}{\partial c} dc + \frac{\partial x}{\partial c_1} dc_1$$

and from $Q(x) = 0$ we derive

$$Q'(x) \frac{\partial x}{\partial c} + \frac{\partial Q(x)}{\partial c} = 0$$

and hence

$$Q'(x) \frac{\partial x}{\partial c} + 2(c_1 x + c) = 0$$

that is

$$Q'(x) \frac{\partial x}{\partial c} + 2\psi(x) = 0$$

It follows that

$$\Sigma \frac{1}{\psi(x)} \frac{\partial x}{\partial c} = -2\Sigma \frac{1}{Q'(x)}$$

and this is 0.

Similarly

$$Q'(x) \frac{\partial x}{\partial c_1} + 2x\psi(x) = 0$$

and hence

$$\Sigma \frac{1}{\psi(x)} \frac{\partial x}{\partial c_1} = -2x\Sigma \frac{1}{Q'(x)} = 0$$

Questions

It should be possible to make purely algebraic sense of the previous sections. The main problem is: how to be sure that the polynomial $Q(x)$ is separable? Intuitively, it should be, since it depends on indeterminate coefficients, but this should be seen *a priori*, without having to do the computation of some discriminant. Also, Abel says that once the addition formula is established, "it is clear" that it holds also in the case where some of the x_1, x_2, x_3 are equal. Is that so clear?

In the case $y^2 = x - x^3$ this claims the following result, that in the extension $k(x_1, y_1, x_2, y_2)$ if we define

$$z = -\frac{c^2}{x_1 x_2} = \frac{x_1 x_2 (1 + x_1 x_2)^2}{(x_1 y_2 + x_2 y_1)^2}$$

and

$$c_1 = \frac{y_1 - y_2}{x_1 - x_2} \qquad c = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

then we shall have

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dz}{c + c_1 z} = 0$$

and also

$$(c + c_1 z)^2 = z - z^3$$

These are non trivial purely computational properties.

There should be a similar interpretation in the hyperelleptic case considering the extension $k(x_1, y_1, x_2, y_2, x_3, y_3)$.

In the case $y^2 = x - x^3$ the differential form $\frac{dx}{y}$ is holomorphic. Indeed the places where $y = 0$ are

1. $(0, 0)$ with parameter y and $x(1 - x^2) = y^2$ so that $(1 - 3x^2)dx = 2ydy$ and $\frac{dx}{y} = \frac{2dy}{1-3x^2}$
2. $(1, 0)$ with parameter y and $\frac{dx}{y} = \frac{2dy}{1-3x^2}$
3. $(-1, 0)$ with parameter y and $\frac{dx}{y} = \frac{2dy}{1-3x^2}$

How can we connect this fact with the fact that we find 0 when we do the sum of this differential form over the points of intersection of the curve with another one?

References

[1] N.H. Abel Extraits de quelques lettres a Crelle Oeuvres complètes.