Remark on the history of Boolean models

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Introduction

In [4], D. Scott refers to an early presentation of Boolean-valued model, due to P. Lagerström, which is described in [2]. What is presented however in this paper is only the statement of the result, and not the proofs. What we present in this note are these proofs, extracted from Church's original notes (available at the Princeton Firestone library). Interestingly, what is missing is a satisfactory treatment of the axiom of extensionality for propositions.

1 Axiom of Choice

The axiom 11^α has the form

$$\exists t. \forall f. \forall x.\ f\ x \to f\ (t\ f)$$

We show the indepence of this from the axioms 1-8 and $10^{\alpha\beta}$.

The idea of Lagerström is to use a *non atomic complete* Boolean algebra for the interpretation of the type of proposition o. Using completeness, it is clear how to interpret universal and existential quantifications, in such a way that the axioms 1-8 and $10^{\alpha\beta}$ are satisfied.

Let t be fixed and let a the value of the formula

$$\forall f. \forall x.\ f\ x \to f\ (t\ f)$$

We want to show that $a = 0$. We show that for any m we have $a \leqslant m$ or $a \leqslant 1 - m$. For this, we consider a function taking only the values m and $1 - m$ and no other values. There are then two cases for such a function f

1. $f\ (t\ f) = m$, in this case $\forall x.\ f\ x \to f\ (t\ f)$ takes the value m and so $a \leqslant m$
2. $f\ (t\ f) = 1 - m$, in this case $\forall x.\ f\ x \to f\ (t\ f)$ takes the value $1 - m$ and so $a \leqslant 1 - m$

If the Boolean algebra is *non atomic* this implies $a = 0$ as desired.

2 Axiom of Description

The formula is

$$\exists t. \forall f. \forall x.\ (\forall y.\ f\ y \to x = y) \wedge f\ x \quad \to f\ (t\ f)$$

As before, we let a be the value of

$$\forall f. \forall x.\ (\forall y.\ f\ y \to x = y) \wedge f\ x \quad \to f\ (t\ f)$$

and we show that we have $a \leqslant m$ or $a \leqslant 1 - m$ for any Boolean value m. This time, we take for f a function taking the values m and $1 - m$ exactly once, and the value 0 otherwise.

Conclusion

The notes of Church show that he was looking for a model which validates the axiom of description and does not validate the axiom of choice. This has been achieved in [1].

References

[1] P. Andrews. General Models, Description and Axiom of Choice in Type Theory. The Journal of Symbolic Logic 37, 1973.

[2] A. Church. Non normal truth-tables for the propositional calculus. Boletin de la Sociedad Matematic Mexicana, vol. I, 1953.

[3] A. Church. A formulation of the simple theory of types. The Journal of Symbolic Logic, 1940.

[4] D. Scott. Preface of Bell's book on Boolean-valued models. 1977.