A remark on contractible family of types

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Introduction

This note complements the reference [1]. We give a general filling property for contractible types. We use then this property to give an interpretation of the axiom of univalence.

1 Nominal sets

We try to work with a nominal set presentation of cubical sets [2]. Any element u depend on at most finitely many element. Also given any element u and any name x we can form $u(x=0)$ and $u(x=1)$ and $u(x=y)$ if y is another name. We have some conditions on these operations that are listed in [2]. We write $x\#u$ to express that u does not depend on x. This is equivalent to $u(x=0)=u$ and equivalent to $u(x=1)=u$.

Some elements represent types and we have a typing relation $u:A$. This relation is invariant under substitution so that $u:A$ implies $u(x=i):A(x=i)$ for $i=0,1$ and implies $u(x=y):A(x=y)$.

It seems very interesting that unique choice can be used at the meta-level. So given u a finite set of names I containing the ones on which u depends is not uniquely determined in general, but we can introduce such a set to build another element, as long as this element will not depend on the choice of this finite set I. This will often be used by chosing a name such that all given objects are independent of this name, and building then a new object which is independent of the choice of this name. (Examples of this are given below.)

We work only with types satisfying the Kan filling conditions. This means the following. If A is a type and if we are given a finite set of names x, J and a corresponding *open box* in A that is a compatible family of elements $u_{yb} : A(y=b)$ for (y,b) in $O^i(x,J)$ with $i=0$ or $i=1$ then we can find u in A such that $u(y=b)=u_{yb}$ for all y,b.

This definition is more general than the usual one, since it may be that A does depend on some names.

Given this generalization we don't need the notion of Kan fibration in this setting! A dependent type over A will be given by a function B such that $B\ u$ is a type for $u:A$. Using that application commutes with substitution, we recover the usual notion of Kan fibration.

2 Contractible types

We assume that A is a type such that we have $c:A$ and a function $t\ u : \mathsf{Id}_A\ c\ u$ for any $u:A$. If this is the case we show that A satisfies the following strong filling property. If $u_{yb} : A(y=b)$ is a family of compatible elements for y in J and $b=0,1$ then we can find $u:A$ such that $u(y=b)=u_{yb}$ for all y,b. The precondition is that u_{yb} is independent of y.

For this we start to choose a name x such that A,s,t are independent of x. We then define the following open box $v_{x0}=c$ and $v_{yb} : c(y=b) \rightarrow_x u_{yb}$ is obtained by taking

$$\langle x\rangle v_{yb} = t(y=b)\ u_{yb}$$

Here we have used that $t : \Pi\ A\ (\mathsf{Id}_A\ c)$ and hence that we also have

$$t(y=b) : \Pi\ A(y=b)\ (\mathsf{Id}_{A(y=b)}\ c(y=b))$$

so that

$$t(y=b)\ u_{yb} : \mathsf{Id}_{A(y=b)}\ c(y=b)\ u_{yb}$$

The family c, v_{yb} is an open box since c is independent of x and v_{yb} is independent of y. Composing this open box in A gives us an element u in A such that $u(y=b) = y_{yb}$ for all y, b. Furthermore this element is independent of the choice of x.

3 Application to equivalence

If $f : A \to B$ then for $b : B$ the type

$$\Sigma a : A.\mathsf{Id}_B\ (f\ a)\ b$$

is the *fiber* Fib b of f at b. The map f is an *equivalence* iff all fibers Fib b are contractible.

We choose x such that A, B, f are independent of x. We define $E : A \to_x B$. An element of $E(A, B, f, x)$ is a pair (a, v) where $a : A$ is indenpendent of x and $v : B$ is such that $v(x = 0) = f\ a$. We define $(a, u)(x = 0) = a$ and $(a, u)(x = 1) = u(x = 1)$, while we have $(a, u)(y = b) = (a(y = b), u(y = b))$ if $y \neq x$. We also define

$$E(A, B, f, x)(x = 0) = A \qquad E(A, B, f, x)(x = 1) = B$$

$$E(A, B, f, x)(y = b) = E(A(y = b), B(y = b), f(y = b), x)$$

We show that this type satisfies the Kan filling property.

For this we assume given an open box of directions z, J in $E(A, B, f, x)$.

There are two cases $x = z$ and $x \neq z$.

If $x \neq z$ and x not in J we first add x to J. For this we look at the given open box (a_{yb}, v_{yb}) with $v_{yb}(x = 0) = f(y = b)\ a_{yb}$.. The completion of a_{yb} in A gives an element a_{x0}. The completion of $v_{yb}(x = 1)$ in B gives an element b_{x1}. The completion of $f\ a_{x0}$ and v_{yb} and b_{x1} gives an element v such that $v(x = 1) = b_{x1}$ and $v(x = 0) = f\ a_{x0}$. The element (a_{x0}, v) is then the required element in $E(A, B, f, x)$. If x is in J we are already given an element a_{x0} and b_{x1} and we proceed similarly by completing in B the family $f\ a_{x0}$ and b_{x1} and v_{yb}.

If $x = z$ there are two sub-cases. One sub-case is where we are given $b_{x1} : B$ independent of x and (a_{yb}, v_{yb}) for y in J. The element $(a_{yb}, \langle x\rangle v_{yb})$ is then an element of $(\mathsf{Fib}\ b_{x1})(y = b)$. By the result of the previous section we can find an element (a, p) in Fib b_{x1}. If we write $p = \langle x\rangle v$ we obtain the element (a, v) such that $v(x = 1) = b_{x1}$ and $v(x = 0) = f\ a$ and (a, v) is the required element in $E(A, B, f, x)$.

The other sub-case is where we are given a_{x0} and a family of elements (a_{yb}, v_{yb}) for y in J. If we fill in B the open box $f\ a_{x0}$ and v_{yb} we obtain an element v in B such that $v(x = 0) = f\ a_{x0}$ and (a_{x0}, v) is the required element in $E(A, B, f, x)$.

4 How do we interpret univalence?

A priori, we have only so far given an interpretation of

$$\sigma_{A,B} : \mathsf{Equiv}\ A\ B \to \mathsf{Id}_{\mathsf{Type}}\ A\ B$$

This interpretation however satisfies strong properties that implies that this defines an inverse of the canonical map

$$\tau_{A,B} : \mathsf{Id}_{\mathsf{Type}}\ A\ B \to \mathsf{Equiv}\ A\ B$$

The first strong property is that the equality $E(A, A, f, x)$ is equal to $A : A \to_x A$ if f is the identity. For this we define a new type $S(A, x, y)$ given $y \neq x$ such that

$$S(A, x, y)(y = 0) = E(A, A, f, x) \qquad S(A, x, y)(y = 1) = A$$

An element of $S(A, x, y)$ is of the form $\langle u\rangle$ where u is an element of A but we take

$$\langle u\rangle(y = 0) = (u(y = 0)(x = 0), u(y = 0)) \qquad \langle u\rangle(x = i) = u(x = i) \qquad \langle u\rangle(y = 1) = u(y = 1)$$

The second property is that the composition

$$\mathsf{Id}_{\mathsf{Type}}\ A\ B \to \mathsf{Equiv}\ A\ B \to (A \to B)$$

sends a path $E : A \to_x B$ to the transport function $A \to B$ of E. On the other hand, the transport function of $E(A, B, f, x)$ is the function f itself. Since

$$\mathsf{Equiv}\ A\ B = (\Sigma f : A \to B)\mathsf{IsEquiv}\ A\ B\ f$$

and $\mathsf{IsEquiv}\ A\ B\ f$ is a proposition, this shows that σ and τ are inverse maps.

References

[1] M.Bezeem, Th. Coquand and S. Huber. A model of type theory in cubical sets. Preprint, 2013.

[2] A. M. Pitts. An Equivalent Presentation of the Bezem-Coquand-Huber Category of Cubical Sets. Manuscript, 17 September 2013.