Constructive Kan Fibration

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Thierry Coquand and Simon Huber

University of Gothenburg

Context

Univalent Foundation (Voevodsky)

Based on a model of MLTT where *types* are interpreted as *homotopy types*

An equality proof of a_0 and a_1 is interpreted as a path from a_0 to a_1

This model validates the MLTT laws of identity types but also *extensionality* and the fact that *isomorphic types* are *equal*

This implies that *isomorphic structures* are equal

Context

There are different ways to represent homotopy types

The most usual is as (Kan) simplicial sets, presheaves over the category of finite non empty linear poset and monotone maps

However, all these models are justified in a *classical* meta-theory, and cannot be used as an implementation of type theory + univalence

The existing implementations add univalence as an axiom, without a computational justification

Contribution

We provide a new class of models of MLTT + identity types with types having non trivial equality (like the groupoid model)

These models are described in a constructive meta-theory, and can be used to get an implementation of MLTT + identity types *justifying the axiom of extensionality*

We think that one particular model justifies also an operator of *propositional reflection* (implementation in progress)

This model should also justifies the axiom of univalence in an effective way (implementation in progress)

How to model extensional identity types

We work in a constructive world of "sets" which models MLTT *without* identity types

We have a notion of dependent family of sets $A\rho,\ \rho : \Gamma$ and we can form a new set $\Gamma.A$ the set of pairs ρ, u with $\rho : \Gamma,\ u : A\rho$ and then if $A\rho,\ \rho : \Gamma$ and $B(\rho, u),\ \rho, u : \Gamma.A$ we can form $(\Pi\ A\ B)\rho,\ \rho : \Gamma$ and $(\Sigma\ A\ B)\rho,\ \rho : \Gamma$

We work in a constructive world of "sets" which models MLTT *without* identity types

How to model extensional identity types

In this world, we select a special set $\mathbf{I}$ with two distinct elements 0 and 1

This set is otherwise arbitrary (and does not need to have a decidable equality; indeed in the interesting examples, $\mathbf{I}$ will not be discrete)

The idea is then to introduce a new type $\mathsf{Path}_A\ a_0\ a_1$ interpreted as the set of "formal paths" $\omega : \mathbf{I} \to A$ such that $\omega\ 0 = a_0$ and $\omega\ 1 = a_1$

This type should satisfy the laws of the identity type

How to model extensional identity types

A first remark is that the axiom of extensionality is valid for this interpretation

$$\frac{\Gamma.A \vdash p : \mathsf{Path}_B\ b_0\ b_1}{\Gamma \vdash \mathsf{ext}\ p : \mathsf{Path}_{\Pi\ A\ B}\ (\lambda\ b_0)\ (\lambda\ b_1)}$$

This is valid for any dependent family $A\rho,\ \rho : \Gamma$ and $B(\rho, u),\ \rho, u : \Gamma.A$ and for any set $\mathbf{I}$ with two elements

How to model extensional identity types

It is also simple to interpret the axiom of reflexivity, if $a : A$ the constant map $\mathsf{ref}_a : \mathbf{I} \to A,\ \mathsf{ref}_a = \lambda i.a$ is an element in $\mathsf{Path}_A\ a\ a$ since $\mathsf{ref}_a\ 0 = \mathsf{ref}_a\ 1 = a$

What is missing in order to get a model of identity type in this way?

How to model extensional identity types

What is missing is Leibniz's Law of indiscernibility of identicals

So we need at least that if $A\rho$ is a dependent family over $\rho : \Gamma$ and $\alpha : \mathsf{I} \rightarrow \Gamma$ then we have a function $A\alpha(0) \rightarrow A\alpha(1)$ and a function $A\alpha(1) \rightarrow A\alpha(0)$

There is no reason that an arbitrary family satisfies this condition

How to model extensional identity types

In order to get a model, we need to find a class of dependent families which satisfies this condition that we can substitue “equals” for “equals” and which furthermore is stable under the operation of dependent sums and dependent products

How to model extensional identity types

It is rather direct to check that such a condition is

for any map $\alpha : \mathbf{I} \to \Gamma$ and any elements $i\ j : \mathbf{I}$ we have a map $A\alpha(i) \to A\alpha(j)$

We call this the *pointwise extension property*

Furthermore this map should be the identity if $i = j : \mathbf{I}$ (extension) and also if α is a constant function, in order to get an interpretation of the computation rule for identity type

(Fibrations having this kind of extension property for $i = 0$ are called Hurewicz regular fiber spaces in algebraic topology)

How to model extensional identity types

We want the model to be closed under identity type as well

So $\mathrm{Path}_A\ a_0\ a_1$ as a family over $(a_0, a_1) : A \times A$ should have the pointwise extension property

We get in this way a further extension condition on $A\rho,\ \rho : \Gamma$, which now involves maps $\mathbf{I}^n \to \Gamma$ for arbitrary n. It is then possible to check that this condition is stable under dependent product and sums

This condition is similar to the cubical version of the Kan extension condition. The main difference is that what closure under dependent product and sum can be done in a constructive meta-theory

How to model extensional identity types

The extension condition is the following where D_n^i is the subset of $\mathbf{I}^n$ of elements $i_1, \ldots, i_n$ such that $i_n = i$ or one of $i_p,\ p < n$ is 0 or 1

Given $\sigma : \mathbf{I}^n \to \Gamma$ any partial section in

$$\prod_{I:D_n^i} A\sigma(I)$$

extends to a section in

$$\prod_{I:\mathbf{I}^n} A\sigma(I)$$

How to model extensional identity types

To get a model of (Christine's version of) the elimination rule for identity, we need to check that, for any type A and any element $a : A$, in the interpretation of

$$\sum_{x:A} \mathrm{Path}_A\ a\ x$$

there is a path from the element $(a, 1_a)$ to any element (x, ω)

This follows from the extension condition for $n = 2$

Examples

If we start from the usual set theoretic model, and $\mathbf{I} = \{0, 1\}$ then we do not get an interesting model

If $A\rho,\ \rho : \Gamma$ has the pointwise extension property then $A\rho$ has to be uniformely inhabited as soon as one fiber $A\rho$ is inhabited, so we cannot interpret non trivial dependent families

Examples

We get interesting models if we start from *presheaf models*:

1. presheafs over non empty finite linear orders and monotone maps (constructive version of *Kan simplicial sets*)
2. presheafs over the free cartesian category with one object $\mathbf{I}$ and two maps $0\ 1 : 1 \to \mathbf{I}$

We have implemented a type-checker for the second model

Presheaf models

A presheaf model on a category $\mathcal{C}$ has itself a concrete description

We write $X, Y, Z, \ldots$ the objects of $\mathcal{C}$ and $f, g, h, \ldots$ the maps of $\mathcal{C}$. If $f : X \to Y$ and $g : Y \to Z$ we write gf the composition of f and g. We write $1_X : X \to X$ or simply $1 : X \to X$ the identity map of X. Thus we have $(fg)h = f(gh)$ and $1f = f1 = f$.

Presheaf models

A context is interpreted by a presheaf Γ: we have a set $\Gamma(X)$ for each object X and an operation $\rho f : \Gamma(Y)$ if $\rho : \Gamma(X)$ and $f : Y \rightarrow X$

We should have $\rho 1 = \rho$ and $(\rho f)g = \rho(fg)$

A type $\Gamma \vdash A$ is interpreted by a family of sets $A\rho$ for $\rho : \Gamma(X)$. We have an operation $uf : A(\rho f)$ if $u : A\rho$ and $f : Y \rightarrow X$

We should have $u1 = u$ and $(uf)g = u(fg)$

Presheaf models

An element $\Gamma \vdash a : A$ is interpreted as a section $a\rho : A\rho$ if $\rho : \Gamma(X)$

We should have $(a\rho)f = a(\rho f) : A(\rho f)$ if $f : Y \to X$

Examples

The objects of the second model can also be described as *covariant* functors on the category of finite sets and maps $I \to J + 2$

One possible syntactic description is that we add in a free way a type $\mathbf{I}$ with two elements $0\ 1 : \mathbf{I}$. Intuitively an element $\rho : \Gamma(K)$ is an environment depending on an element in $\mathbf{I}^K$

The only definable maps $\mathbf{I} \to N$ are the constant maps and we can model non trivial dependent types over N

An element depending on n variables in $\mathbf{I}$ can be thought of as a hypercube so we get a variation of the cubical set model

Example

What do we get? An implementation of MLTT with *extensional* identity types which has the canonicity property

To be done: propositional reflection and univalence

Example

Propositional reflection: we add a new operation $A*$ such that

1. $A*$ is always a proposition (i.e. of hlevel 1)
2. we have $A \to A*$
3. if P is a proposition and $A \to P$ then $A* \to P$

We can then define $\exists\ A\ B = (\Sigma\ A\ B)*$ and $\{A \mid B\} = \Sigma\ A\ B*$

It is possible to interpret such an operation in the simplicial set model and in the cubical model

In both cases, we take $A * \rho$ for $\rho : \Gamma(X)$ to be the set of all family $a_f : A(\rho f)$ for $f : 1 \to X$ global element of X