A Topos Theoretic Fix Point Theorem

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Introduction

We present a fix point theorem that can be seen as an intuitionistic alternative to Bourbaki's lemma used in his presentation of Zorn's lemma [3]. Here however intuitionistic is taken in a generalized sense since our proof uses impredicativity. We present then an application of this lemma to the semantics of Frege Structure and of Type Theory, and compare it to S. Allen's justification of this semantics.

1 A Fix Point Theorem

Theorem: Let D be a conditional sup semi-lattice, with a least element $\perp$, such that any directed subset has a least upper bound. Let $f : \mathsf{D} \rightarrow \mathsf{D}$ be a monotone map. Then, f has a fix point.

This statement differs from Bourbaki's lemma in the assumption that D has conditional binary sups, and that f is monotone instead of satisfying $x \leq f(x)$. However, we will consider subsets of D over which $x \leq f(x)$ holds. Furthermore, in the application we present, both results can be used.

The main point here is that this theorem can be proved without using the law of excluded middle, which may be of interest in the framework of topos theory [4]. We discuss below the foundational problem raised by the use of impredicativity in the applications of this result.

Proof: We consider the subset $A \subseteq \mathsf{D}$ generated by the clauses

- $\perp \in \mathsf{D}$,
- $f(x) \in A$ if $x \in A$,
- $x \vee y \in A$ if $x, y, z \in A$ and $x \leq z,\ y \leq z$,
- $\bigvee X \in A$ if $X \subseteq A$ is directed.

It can be shown by induction that A is a directed subset of D. Also by induction, we can see that $x \leq f(x)$ if $x \in A$. Hence, it has a least upper bound $x_0 = \bigvee A$, which belongs to A by the last clause, and is thus the bigger element of A. Hence we have $f(x_0) \in A$ and $x_0 \leq f(x_0),\ f(x_0) \leq x_0$. $\square$

2 Application to the semantics of Type Theory

Let X, Y be given set, and P the following poset: the elements are partial functions $f : \mathbf{Dom}(f) \rightarrow Y$, with $\mathbf{Dom}(f) \subseteq X$, and $f \leq g$ iff $\mathbf{Dom}(f) \subseteq \mathbf{Dom}(g)$ and $f(x) = g(x)$ for all $x \in \mathbf{Dom}(f)$ (that is, g is an extension of f.

Proposition: P is a conditional sup-semi lattice, with at least element $\perp$ the empty function $\emptyset \rightarrow Y$.

Following [1], we can define an operator $T : P \rightarrow P$ and apply the theorem to build a model of Type Theory. There are still some analysis to be done, but it seems that this approach is essentially impredicative, though the impredicativity used may be of a different nature than the one used in the construction of S. Allen [2].

ALTERNATIVE: use Brouwer's ordinals, and show the existence of a fixed point for T in a type theory with generalised inductive definition. Is this possible???

References

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[3] N. Bourbaki. Théorie des Ensembles, Livre I, Chapitre III, Exercice 6. Hermann, Ed. 1956

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