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A system of ordinal notations with natural representations in the second order λ-calculus

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Abstract

`Sort out this boldface; Abstract and boldsymbol`

I describe a system of ordinal notations which have straight forward and direct representations in the second order λ-calculus $\boldsymbol{\lambda 2}$. `I believe these ordinals cofinally exhaust those representable in` $\boldsymbol{\lambda 2}$.

`[Held in 100.../41-.../000-bit....   Last changed September 12, 1994]`

1 Introduction

`[Held in 100../41../100-bit...   Last changed September 12, 1994]`

There are now many type systems (formal systems) in use in Mathematics and Theoretical Computer Science. Some are designed for quite specific purposes. Amongst these are the many systems used to analyse various aspects of number theory, analysis (second order number theory), and restricted versions of set theory. The items [6], [7], [8], [12], [13], [27], [28], [29], [30], [31], [32] give a fair sample of this material. Other systems are designed as formatting devices for general purpose use in which more specific systems can be formalized. Amongst these are the Pure Type Systems [3]; the family of Logical Frameworks [19]; the Calculus of Constructions [20]; and Martin-Löf type theory [24], [25]. Many of these have implemented versions such AUTOMATH [5], Nuprl [9], Coq [11], HOL [18], LEGO [22], and Isabelle. The article [26] is a nice survey of this topic.

Given this plethora of systems how should we compare the strengths and capabilities of two particular examples? We could present the two systems in the same formalism and then look for a direct comparison (in terms of a suitable notion of embedding or translation). This may seem the best thing to do, but what happens if we change (even slightly) one of the systems or want to compare a given system with a family of systems.

A more general technique for comparison is to attach to each system a measure of 'amount of interesting information' coded by the system. These measures are chosen so that a direct comparison between the assigned values is relatively straight forward. In this context the most common measure used is the class of number theoretic functions represented in a system. When we do this we expect the assigned measure to be something like the class of

- rudimentary functions
- primitive recursive functions
- those functions provable total in 1^{st} order number theory
- those functions provable total in 2^{nd} order number theory

$\vdots$

or various well defined subclasses of these.

Many of these classes are generated by a common iterative process, and the ordinal number of steps required to generate the whole class is a cruder, but still important, measure of information. When we do this we expect ordinals such as

- the least infinite ordinal ω
- the least critical ordinal ϵ_0
- the least strongly critical ordinal Γ_0

$\vdots$

- the Howard ordinal

$\vdots$

to be the assigned measure.

To determine such an ordinal measure we need to find both an upper bound and a lower bound and hope these two bounds agree (otherwise we have merely sandwiched the value in an interval of ordinals). In general, these two bounds are found using quite different methods. The upper bound will be computed using some operational behaviour of the system (such as a proof of strong normalization).

A lower bound can be obtained by representing ordinals (or structures which code ordinals) directly in the system. This representation should interact smoothly with the representation of number theoretic functions in such a way that ordinal iteration is captured in a fairly direct fashion. Furthermore, these representations of ordinals and functions should reflect some of the structure of the system. An ad hoc representation may work by accident rather by virtue of some inherent property of the system.

In the paper I will produce a lower bound Δ for the ordinals represented in the second order λ-calculus $\boldsymbol{\lambda 2}$, (also called the polymorphic λ-calculus and system F). This ordinal Δ will be constructed as the limit of a sequence of ordinals

$$\Delta[0] < \Delta[1] < \Delta[2] < \cdots < \Delta[r] < \cdots$$

which is closely related to the type structure of $\boldsymbol{\lambda 2}$. Each ordinal $\Delta[r]$ will be represented in $\boldsymbol{\lambda 2}$ in a rather direct fashion. It turns out that

$$\begin{aligned} \Delta[0] &= \omega \\ \Delta[1] &= \epsilon_0 \\ \Delta[2] &= \epsilon_{\epsilon_{\epsilon_{\cdot_{\cdot_{\cdot}}}}} \\ \Delta[3] &= \Gamma_0 \end{aligned}$$

and the gap between $\Delta[r]$ and $\Delta[r+1]$ increases in order of magnitude as r increases. I believe the ordinal

$$\Delta = lim(\Delta[r] \mid r < \omega)$$

is also an upper bound for the complexity of $\boldsymbol{\lambda 2}$. However, I do not present a proof of this.

The problem of determining the complexity of $\boldsymbol{\lambda 2}$ is not new, but it is usually done in terms of the corresponding class of number theoretic functions (those which are provable total in second order number theory). It is unusual to see a direct representation of the ordinals.

In that excellent little book [16] the complexity of various systems is measured through the represented functions. Ordinals are hardly mentioned (either as formal objects or informal tools). In [15] there is a lot of information about ordinals associated with systems, but almost always these ordinals are not represented directly within the system.

Much of the paper [14] is about the complexity of $\boldsymbol{\lambda 2}$, and sections 5.5 and 5.6 can be seen as a representation of the ordinal ϵ. However, this representation is not done explicitly, and the more general problem is not considered.

One place where the direct representation of ordinals is described is at the end of section 4 of that useful survey [21]. This material, which is based on the work of Coquand [10], has had a lot of influence on the results presented here.

In essence this paper is a description of how $\boldsymbol{\lambda 2}$ can be used to construct a natural system of ordinal notations. It should, therefor, be related to the many other systems of ordinals notations devised. It wouldn't be constructive (in the non-technical sense) to survey the literature on this topic. The best places to begin are the accounts [27], [28], [29] and work backwards from the papers cited there. I should mention also the papers [1] and [23] which have given me a lot of food for thought. The paper [17] includes a very readable introduction to ordinal notations (but doesn't go as far as needed for our purposes).

What is the idea underlying this and almost all systems of ordinal notations?

Let $\mathbb{O}$ be an initial stretch of ordinals which contains all ordinals we may want to use, but is not too long as to be excessive. If you want to be precise think of $\mathbb{O}$ as the stretch of countable ordinals. We will behave as though $\mathbb{O}$ were the only existing ordinals. Thus, for instance we say

the ordinal α	instead of	the ordinal $\alpha \in \mathbb{O}$
the set A of ordinals	instead of	the subset $A \subseteq \mathbb{O}$

etc. This, of course, is nothing more than a convenience; it doesn't imply anything about a perceived non-existence of other ordinals, we just don't need the others here.

We want to name as many ordinals (in $\mathbb{O}$) as possible using a small array of gadgets. We can use the zero ordinal and the successor function $\alpha \mapsto \alpha + 1$. We can describe a limit ordinal λ as the limit of an ascending sequence

$$\lambda[0] \leq \lambda[1] \leq \cdots \leq \lambda[r] \leq \cdots$$

of strictly smaller ordinals provided the function

$$\lambda[\cdot] : \mathbb{N} \longrightarrow \mathbb{O}$$

can be described. This function $\lambda[\cdot]$ is a chosen fundamental sequence for λ, and it is precisely the selection of these sequences which is one of the main problems when designing a system of ordinal notations.

How can we generate notations?

Let

$$\mathbb{O}' = \mathbb{O} \to \mathbb{O}$$

be the type of ordinal functions. Suppose also we have constructed a function

$$Next : \mathbb{O}'$$

which when applied to an ordinal ζ produces a larger ordinal $Next\,\zeta$ and which we can take as a notation for this ordinal (once we have a notation for ζ and $Next$). By iterating this process we obtain

$$\zeta,\ Next\,\zeta,\ Next^2\zeta = Next(Next\,\zeta),\ Next^3\zeta = Next(Next^2\zeta),\ \ldots$$

and, after some preparation, we can take this into the transfinite

$$Next^\alpha\zeta$$

for a suitable stretch of ordinals α. Thus, given a notation for α and ζ, we produce a notation for a much larger ordinal. The simplest example of this uses the function $Next\zeta = \omega^\zeta$.

The choice of ζ and $Next$ ensure that $\alpha \leq Next^\alpha\zeta$ and initially α is strictly smaller than $Next^\alpha\zeta$. However, eventually we meet an ordinal ν satisfying

$$\nu = Next^\nu\zeta \tag{1}$$

and at this point the process is incapable of producing any more notations.

How can we continue?

Let

$$\mathbb{O}'' = \mathbb{O}' \to \mathbb{O}'$$

be the type of operators which convert ordinal functions into ordinal functions. I will describe a particular operator

$$\Box : \mathbb{O}''$$

which, when supplied with a suitable pair

$$Next : \mathbb{O}' \quad , \quad \zeta : \mathbb{O}$$

will return an ordinal

$$\nu = \Box Next\zeta$$

which is the least solution of (1). We then use '$\Box Next\zeta$' as a notation for this ν.

This process is iterated to produce a sequence

$$\zeta,\ Next\zeta,\ \Box Next\zeta,\ \Box^2 Next\zeta,\ \Box^3 Next\zeta,\ \ldots,\ \Box^\alpha Next\zeta,\ \ldots,$$

of larger and larger ordinals. As before this closes off at the least solution of

$$\nu = \Box^\nu Next\zeta \tag{2}$$

and at this point we need to enrich further the allowable gadgetry.

Let

$$\mathbb{O}''' = \mathbb{O}'' \rightarrow \mathbb{O}''$$

be the type of constructors on operators. I will produce an inhabitant of $\mathbb{O}'''$ which when supplied with $\Box$, $Next$ and ζ will return the least solution of (2). You can guess what will happen next.

Consider the sequence

$$\mathbb{O}^{(0)},\ \mathbb{O}^{(1)},\ \mathbb{O}^{(2)},\ \ldots,\ \mathbb{O}^{(r)},\ \ldots$$

of types generated by

$$\mathbb{O}^{(0)} = \mathbb{O} \quad , \quad \mathbb{O}^{(r+1)} = \mathbb{O}^{(r)\prime} = \mathbb{O}^{(r)} \rightarrow \mathbb{O}^{(r)}$$

for all $r < \omega$. I will produce a sequence of inhabitants

$$Next : \mathbb{O}',\ \Box : \mathbb{O}'',\ \boxed{0} : \mathbb{O}''',\ \boxed{1} : \mathbb{O}^{(4)},\ \ldots,\ \boxed{r} : \mathbb{O}^{(r+3)},\ \ldots$$

each of which extracts the least solution of a certain fixed point equation. These inhabitants will have a comparatively straight forward representation in $\boldsymbol{\lambda 2}$. Furthermore, the whole sequence will be constructed in a uniform fashion, but at first sight it seems that this uniformity can not be captured in $\boldsymbol{\lambda 2}$ (because it makes use of a type constructor not available in the system).

Setting

$$\begin{array}{lcl}
\Delta[0] & = & \omega \\
\Delta[1] & = & Next\omega \\
\Delta[2] & = & \Box Next\omega \\
 & \vdots & \\
\Delta[r+3] & = & \boxed{r} \cdots \boxed{0} \Box Next\omega \\
 & \vdots &
\end{array}$$

produces the ordinal stratification of $\boldsymbol{\lambda 2}$.

(The indexing of $Next$, $\Box$, $\boxed{0}$, $\boxed{1}$, ... might look a little eccentric, but it is not without reason. It is done partly for presentational reasons, but mainly to smooth out some of the initial stages of the whole construction.)

The remainder of this paper is divided into 8 further sections as follows.

Section 2 sets down the basic concepts used throughout the paper. In particular, there is a discussion of how the higher order inhabitants of $\mathbb{O}^{(k+1)}$ (for all $k < \omega$) may be iterated transfinitely.

Almost all systems of ordinal notations have their origin in the Veblen hierarch $\phi(\cdot,\cdot)$ [33] which enumerates (from below) the critical ordinals $\epsilon_\bullet$ up to the least strongly critical ordinal Γ_0. Section 3 contains a quick review and a rephrasing of the construction of ϕ relativized to an arbitrary normal function $f : \mathbb{O}'$. I write down the obvious construction of the operator Fix : $\mathbb{O}''$ which when supplied with a normal function f and an ordinal ζ, will return, Fix$f\zeta$, the least fixed point of f beyond ζ. This motivates what comes later.

Normal functions are the enumerating functions of closed unbounded sets of ordinals. Section 4 introduces the slightly different notion of a nice function $g : \mathbb{O}'$. These functions are such that for each non-zero ordinal ζ the iteration function

$$\alpha \mapsto g^\alpha \zeta$$

is normal. In particular, for each normal function f, the function Fix$f$ is nice and so

$$\alpha \mapsto (\mathrm{Fix} f)^\alpha \zeta$$

is normal. To prepare for later I construct an operator $\square : \mathbb{O}''$ which produces the least solution of

$$g^\nu \zeta = \nu$$

for a supplied nice $g : \mathbb{O}'$ and non-zero ordinal ζ.

Section 5 relates the construction so far to the standard Veblen hierarchy. Thus we find that

$$\phi_f(1+\beta,\alpha) = (\square^\beta(\mathrm{Fix} f))^{1+\alpha}\omega$$

for each normal $f : \mathbb{O}'$ and ordinals $\alpha, \beta : \mathbb{O}$. The two occurrences of '1 + ' smooth out a couple of discrepancies between the two approaches.

Section 6 contains the first of the two main contributions of this paper. Following on from section 4 I define the notion of a nice inhabitant of each of the types $\mathbb{O}''$, $\mathbb{O}'''$, $\mathbb{O}^{(4)}$, ... We see that the operator $\square : \mathbb{O}''$ is nice, and I extend this to construct nice inhabitants $\boxed{0} : \mathbb{O}'''$, $\boxed{1} : \mathbb{O}^{(4)}$, ... These new operators fit together in a neat way and are used to produce solutions of certain higher order fixed point equations. Furthermore, the constructions of $\square$, $\boxed{0}$, $\boxed{1}$, ... are such that they can be represented in **λ2** in a relatively straight forward manner.

As an interlude, and partial justification, section 7 contains a few calculations which show just how powerful these new operators are.

Section 8 contains the second of the two main contributions of this paper. Natural representations in **λ2** of the operators $\square$, $\boxed{0}$, $\boxed{1}$, ... and the ordinals $\Delta[0]$, $\Delta[1]$, $\Delta[2]$, ... are given. Of course, because of the way these gadgets have been set up, this is now quite easy. However, there are still one or two minor points to deal with.

To conclude, in section 8 I give a selection of open questions, possible further developments, and concluding remarks.

From this summary you can see that the crucial idea is that of a notion of niceness for $\mathbb{O}', \mathbb{O}'', \mathbb{O}''', \ldots$. I can define these straight away along with a notion of niceness for $\mathbb{O}$.

1 DEFINITION. An ordinal $\zeta : \mathbb{O}$ is nice if $\zeta > 0$.

An ordinal function $g : \mathbb{O}'$ is nice if it is monotone and strictly inflationary (in an appropriate sense) when applied to nice ordinals.

Given a notion of niceness for a type $\sigma' = \sigma \to \sigma$, an inhabitant $G : \sigma'' = \sigma' \to \sigma'$ is nice if for each nice $g : \sigma'$ the inhabitant $Gg : \sigma'$ is also nice. ■

The purpose of the first clause of this definition is to eliminate certain abnormalities that can occur in ordinal calculations when 0 is involved.

The second clause is the important one from which all other instances of niceness are derived. This is discussed in detail in section 4.

The third clause of the definition gives us, in turn, a notion of niceness for $\mathbb{O}'', \mathbb{O}''', \mathbb{O}^{(4)}, \ldots$. This is discussed in detail in section 6. In that section a slightly different definition is used because it allows us to derive the required properties more quickly (however, the two notions are the same).

2 Preliminary material

`[Held in 100../41-../200-...   Last changed September 12, 1994]`

As indicated in the introduction $\mathbb{N}$ and $\mathbb{O}$ are, respectively the concrete set of natural numbers and a suitable initial stretch of the ordinals. We are interested in the function space types generated from these base types by

if σ and ρ are types then so is $\sigma \to \rho$

(where ρ, σ range over these types). Eventually we represent these in $\boldsymbol{\lambda 2}$ (using certain polymorphic types $\mathcal{N}$ and $\mathcal{O}$ as base types). Our notation for concrete types closely matches that for $\boldsymbol{\lambda 2}$-types and, in places, I blur the distinction between the two. In particular, the compound type

$$\tau \to \sigma \to \rho$$

is punctuated as

$$\tau \to (\sigma \to \rho)$$

and not as $(\tau \to \sigma) \to \rho$. For each type σ we write

$$\sigma' \quad \text{for} \quad \sigma \to \sigma$$

and we may iterate this construction to produce types σ'', σ''', $\sigma^{(4)}$, $\ldots$, $\sigma^{(r+1)}$, $\ldots$ where

$$\begin{array}{lllll} \sigma'' & = & \sigma' \to \sigma' & = & \sigma' \to \sigma \to \sigma \\ \sigma''' & = & \sigma'' \to \sigma'' & = & \sigma'' \to \sigma' \to \sigma \to \sigma \\ \sigma^{(4)} & = & \sigma''' \to \sigma''' & = & \sigma''' \to \sigma'' \to \sigma' \to \sigma \to \sigma \\ & \vdots & & & \\ \sigma^{(r+1)} & = & \sigma^{(r)} \to \sigma^{(r)} & & \\ & \vdots & & & \end{array}$$

etc. In particular, the concrete types $(\mathbb{O}^{(r)} \mid r < \omega)$ are produced in this way.

Given types σ and ρ, and inhabitants

$$f : \sigma \to \rho \quad , \quad s : \sigma$$

we use the standard notation of λ-calculus and write

$$fs \quad \text{rather than} \quad f(s)$$

for the value of f at s. Furthermore, for

$$f : \tau \to \sigma \to \rho \quad , \quad t : \tau \quad , \quad s : \sigma$$

the compound

$$fts$$

is punctuated as

$$(ft)s$$

(and not as $f(ts)$) to produce a value in ρ.

We use lower case Greek letters $\alpha, \beta, \gamma, \zeta, \lambda, \mu$ to indicate ordinals. In general α, β, γ will be arbitrary ordinals; ζ usually a base ordinal (of some calculation); λ, μ limit ordinals; and ν a solution of a fixed point equation. There will be occasional breaks of this convention. In particular, ρ, σ, and τ are types (either concrete or formal).

We should think of ordinals as templates for long iterations. Given a function $f : \sigma'$ on some type σ, there is no problem in producing finite iterates

$$f^0 = id_\sigma,\ f^1 = f,\ f^2 = f \circ f,\ f^3, \ldots$$

of f. If σ comes furnished with a supremum operation $\bigvee_\sigma$ (which converts subsets of σ into elements of σ) then we can continue further. We set

$$\begin{aligned} f^0 &= id_\sigma \\ f^{\alpha+1} &= f \circ f^\alpha \\ f^\lambda s &= \bigvee_\sigma \{ f^\alpha s \mid \alpha < \lambda \} \end{aligned}$$

for all ordinals α, limit ordinals λ, and $s : \sigma$. We view an ordinal α as a polymorphic gadget which, when supplied with a type σ, a function $f : \sigma'$, and an element $s : \sigma$, will return the value $f^\alpha s$ of the α^{th}-iterate of f at s.

This kind of iteration can be lifted from a base type to higher types. Thus for each type σ we can lift any supremum operation $\bigvee_\sigma$ on σ to a pointwise supremum $\bigvee_{\sigma'}$ on σ' by

$$(\bigvee_{\sigma'} \mathcal{F})s = \bigvee_\sigma \{ fs \mid f \in \mathcal{F} \}$$

for each subset $\mathcal{F}$ of σ' and $s : \sigma$. Using this the above iteration can be rephrased as

$$\begin{aligned} f^0 &= id_\sigma \\ f^{\alpha+1} &= f \circ f^\alpha \\ f^\lambda &= \bigvee_{\sigma'} \{ f^\alpha \mid \alpha < \lambda \} \end{aligned}$$

for all appropriate α, λ and f.

This lifting also allows us to iterate an given $F : \sigma''$ by

$$\begin{array}{lcl} F^0 & = & id_{\sigma'} \\ F^{\alpha+1} & = & F \circ F^{\alpha} \\ F^{\lambda} f & = & \bigvee_{\sigma'} \{F^{\alpha} f \mid \alpha < \lambda\} \end{array}$$

for all $f : \sigma'$ and ordinals α, λ. The last clause here can be rephrased as

$$F^{\lambda} = \bigvee_{\sigma''} \{F^{\alpha} \mid \alpha < \lambda\} \quad \text{or as} \quad F^{\lambda} f s = \bigvee_{\sigma} \{F^{\alpha} f s \mid \alpha < \lambda\}$$

using the lifting $\bigvee_{\sigma''}$ of $\bigvee_{\sigma'}$ to σ'', or the generating supremum $\bigvee_{\sigma}$ on σ.

The type $\mathbb{O}$ carries a standard supremum operation. This can be lifted in steps to pointwise suprema on $\mathbb{O}'$, $\mathbb{O}''$, $\mathbb{O}'''$, ... and so it make sense to write

$$F^{\alpha} f f_l \dots f_1 \zeta$$

for all $F : \mathbb{O}^{(l+2)}$, $f : \mathbb{O}^{(l+1)}$, $f_l : \mathbb{O}^{(l)}, \dots, f_1 : \mathbb{O}'$, and ordinals $\alpha, \zeta : \mathbb{O}$.

All this is unproblematic in the world of concrete sets and function spaces. However, the system $\boldsymbol{\lambda 2}$ doesn't handle arbitrary subsets of types, so we can not expect to capture the properties of arbitrary suprema operations. Instead of suprema $\bigvee_{\sigma} A$ of arbitrary subsets A of σ, we consider only suprema $\bigvee_{\sigma} p$ of ω-sequences $p : \mathbb{N} \to \sigma$ of inhabitants of σ. These can be captured in $\boldsymbol{\lambda 2}$.

In these terms the limit leap f^{λ} of the iteration of a function $f : \sigma'$ is constructed as

$$f^{\lambda} s = \bigvee_{\sigma} \{f^{\lambda[r]} \mid r < \omega\}$$

where $\lambda[\cdot]$ is a chosen fundamental sequence of λ (satisfying $\lambda = \bigvee\{\lambda[r] \mid r < \omega\}$). This, of course, make the iterate f^{α} dependent on the notation for the ordinal α and not just the ordinal itself.

Used in this way we require only suprema of ω-sequences p on a type σ. Such a supremum operation can be lifted to σ'' by

$$(\bigvee_{\sigma'} \{qr \mid r < \omega\}) s = \bigvee_{\sigma} \{qrs \mid r < \omega\}$$

for each $q : \mathbb{N} \to \sigma'$ and $s : \sigma$.

This gives us the suprema operations on $\mathbb{O}', \mathbb{O}'', \mathbb{O}''', \dots$ each of which will be written as $\bigvee$ (without a distinguishing subscript). However, these must be used with some care. The type $\mathbb{O}$ carries a natural comparison $\leq$ relation between its elements which ensures that

$$\bigvee B = \bigvee A$$

for all subsets $B \subseteq A \subseteq \mathbb{O}$ with B cofinal in A. This is precisely why we can compute many suprema by suprema of ω-sequences. These facilities are not readily available in $\mathbb{O}'', \mathbb{O}''', \dots$. To overcome this we restrict our attention to a nice family of inhabitants of each $\mathbb{O}^{(r)}$ which somehow preserve enough of the comparison property of ordinals.

3 The standard Veblen hierarchy

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Recall that an ordinal function $f : \mathbb{O}'$ is **normal** if it is strictly monotone and continuous in the sense that

(sm) $\alpha < \beta \Rightarrow f\alpha < f\beta$

(c) $f(\bigvee A) = \bigvee\{f\alpha \mid \alpha \in A\}$

hold for all ordinals α, β, and all non-empty sets A of ordinals. Such a function is automatically inflationary, i.e. satisfies

(i) $\zeta \leq f\zeta$

for all ordinals ζ. For technical reasons it is convenient to restrict our attention to those normal functions f which dominate the function $\omega^\bullet$ i.e. satisfy

$$\omega^\alpha \leq f\alpha$$

for all ordinals α. This smoothes out the initial stages of various constructions. The least such normal function, i.e. $\omega^\bullet$ itself, is used to generate the Veblen hierarchy. We relativize this construction to an arbitrary normal function.

We are interested in the **fixed points** ν satisfying

$$f\nu = \nu$$

for a given normal function f. These fixed points are easy to generate. Given any $\zeta : \mathbb{O}$ choose some $\zeta^\cdot$ with $\zeta \leq \zeta^\cdot$ and then define a sequence $\nu[\cdot] : \mathbb{N} \to \mathbb{O}$ by

$$\nu[0] = \zeta^\cdot \quad , \quad \nu[r+1] = f(\nu[r])$$

(for $r < \omega$). The monotonicity of f ensures that

$$\zeta \leq \nu[0] \leq \nu[1] \leq \cdots \leq \nu[t] \leq \cdots$$

and the continuity ensures that

$$\nu = \bigvee\{\nu[r] \mid r < \omega\}$$

is a fixed point of f. Furthermore, if μ is any fixed point of f with $\zeta^\cdot \leq \mu$, then $\nu \leq \mu$, so if $\zeta^\cdot$ is not too large then ν is the least fixed point of f beyond ζ. For most ζ, in particular for $\zeta = \omega$, we can take $\zeta^\cdot = \zeta$. If ζ happens to be a fixed point of f then we can take $\zeta^\cdot = \zeta + 1$ or ζ^ω.

The continuity of f ensures that the set of fixed points is closed under suprema, and so is a closed unbounded set. The enumeration of this set is another normal function. We write f' for this function, so

$$\begin{aligned} f'0 &= \text{least fixed point of } f \\ f'(\alpha+1) &= \text{least fixed point of } f \text{ beyond } f'\alpha \\ f'\lambda &= \bigvee\{f'\alpha \mid \alpha < \lambda\} \end{aligned}$$

for all ordinals α and limit ordinals λ.

In the particular case $f = \omega^\bullet$ the function f' enumerates the critical ordinals, $f'\alpha = \epsilon_\alpha$.

The **Veblen hierarchy**

$$\phi_f : \mathbb{O} \times \mathbb{O} \longrightarrow \mathbb{O}$$

on f is generated by

$$\begin{array}{lcl} \phi_f(0,\cdot) & = & f \\ \phi_f(\beta+1,\cdot) & = & \text{enumeration of the fixed points of } \phi_f(\beta,\cdot) \\ \phi_f(\mu,\cdot) & = & \text{enumeration of the common fixed points of } \phi_f(\beta,\cdot) \\ & & \text{for all } \beta < \mu \end{array}$$

for ordinals β and limit ordinals μ. It can be checked that each function $\phi_f(\beta,\cdot)$ is normal and dominates all previous functions in the hierarchy. Furthermore, the function f^+ defined by

$$f^+\alpha = \phi_f(\alpha, 0)$$

is normal, and so provides a base function with which we may repeat the whole process.

In the standard case $f = \omega^\bullet$ the fixed points of f are the critical ordinals, and the fixed points of f^+ are the strongly critical ordinals.

I will rephrase the description of this construction.

Starting from a given normal function f, we iterate the operator $(\cdot)'$ to produce a sequence

$$(f^{(\beta)} \mid \beta \in \mathbb{O})$$

of functions. Thus

$$\begin{array}{lcl} f^{(0)} & = & f \\ f^{(\beta+1)} & = & f^{(\beta)\prime} \\ f^{(\mu)}\zeta & = & \bigvee\{f^{(\beta)}\zeta \mid \beta < \mu\} \end{array}$$

for all ordinals β, ζ and limit ordinals μ. It can be checked that each function $f^{(\beta)}$ is normal. Only the passage across limit stages needs a bit of thought. Note also that

$$f^{(\mu)}\nu = \nu \quad \Leftrightarrow \quad \begin{array}{l} \nu \text{ is a common fixed point} \\ \text{of } f^{(\beta)} \text{ for all } \beta < \mu \end{array}$$

holds for each limit ordinal μ and ordinal ν.

This allows a neater description of the hierarchy ϕ_f. We have

$$\phi_f(1+\beta,\cdot) = f^{(\beta)\prime} = f^{(\beta+1)} \tag{3}$$

for all ordinals β. This is proved by induction on β. For the leap to a limit ordinal μ, letting

e.o.t.c.f.p.o.

stand for

enumeration of the common fixed points of

we have

$$\begin{array}{lcl} \phi_f(1+\mu,\cdot) = \phi_f(\mu,\cdot) & = & \text{e.o.t.c.f.p.o. } \phi_f(\beta,\cdot) \text{ for all } \beta < \mu \\ & = & \text{e.o.t.c.f.p.o. } \phi_f(1+\beta,\cdot) \text{ for all } \beta < \mu \\ & = & \text{e.o.t.c.f.p.o. } f^{(\beta+1)} \text{ for all } \beta < \mu \\ & = & \text{e.o.t.c.f.p.o. } f^{(\beta)} \text{ for all } \beta < \mu \\ & = & \text{e.o.t.c.f.p.o. } f^{(\mu)} \qquad = f^{(\mu)\prime} = f^{(\mu+1)} \end{array}$$

as required.

Let

$$\mathrm{Fix} : \mathbb{O}''$$

be the operator defined by

$$\mathrm{Fix}\, f\zeta \;=\; f^{\omega}\zeta^{\cdot}$$

for all $f : \mathbb{O}'$ and $\zeta : \mathbb{O}$. (Here $\zeta \mapsto \zeta^{\cdot}$ is any suitable inflationary function as described earlier.) Using the previous notation for the construction of fixed points we see that

$$\nu[r] \;=\; f^{r}\zeta^{\cdot}$$

for each $r < \omega$, and hence

$$\nu \;=\; \bigvee\{f^{r}\zeta^{\cdot} \mid r < \omega\} \;=\; f^{\omega}\zeta^{\cdot} \;=\; \mathrm{Fix}\, f\, \zeta$$

to verify the following.

> For each normal function $f : \mathbb{O}'$ and $\zeta : \mathbb{O}$, the ordinal $\nu = \mathrm{Fix}\, f\, \zeta$ is the least fixed point of f beyond ζ.

By our restricted use of normality all the fixed points of f lie beyond ω, hence we have

$$f'\alpha \;=\; (\mathrm{Fix}\, f)^{1+\alpha}\omega \tag{4}$$

for all normal $f : \mathbb{O}'$ and $\alpha : \mathbb{O}$.

Setting $\beta = 0$ in (3) we have

$$\phi_f(1, \alpha) \;=\; (\mathrm{Fix}\, f)^{1+\alpha}\omega$$

for all $\alpha : \mathbb{O}$. We wish to extend this to an iterative description of $\phi_f(1+\beta, \alpha)$ for all ordinals α, β.

4 Nice functions

[`Held in 100../41../400-bit...   Last changed September 12, 1994`]

To continue the development we look at the more general problem of solving

$$g^{\nu}\zeta = \nu \tag{5}$$

for given $g : \mathbb{O}'$ and $\zeta : \mathbb{O}$. Of course to obtain a smooth analysis we need to restrict the given data g, ζ somewhat.

2 **DEFINITION**. An ordinal function $g : \mathbb{O}'$ is nice if it is monotone and strictly inflationary in the sense that

(m) $\alpha \le \beta \Rightarrow g\alpha \le g\beta$

(i) $\zeta < g\zeta$

hold for all nice ordinals α, β, ζ. ■

Do not confuse this notion with that of a normal function. The two notions are related but not the same. To help you distinguish between the two I will use 'f' for a normal function and 'g' for a nice function.

We restrict our attention to equation (5) where the data g, ζ is nice. Before we can do this we need at least one example of a nice function.

3 LEMMA. *For each normal function f the function Fix $f : \mathbb{O}'$ is nice.*

Proof. Let $g = \mathrm{Fix}\, f$. By construction we have $\zeta < g\zeta$ for all non-zero ζ, and hence g is inflationary.

For $0 < \alpha \leq \beta$ let

$$\mu = g\,\alpha \quad , \quad \nu = g\,\beta$$

i.e. μ, ν are the least ordinals satisfying

$$\alpha < \mu = f\mu \quad , \quad \alpha \leq \beta < \nu = f\nu$$

respectively. The built in minimality of μ and ν ensure that $\mu \leq \nu$, and hence g is monotone, as required. ■

It is easy to see that the composite $g \circ h$ of two nice function is nice. Similarly the pointwise supremum of a non-empty collection of nice functions is nice. Thus the non-zero ordinal iterates

$$(g^\alpha \mid \alpha \in \mathbb{O},\ \alpha \neq 0)$$

of a nice function g are all nice.

We need a comparison property.

4 LEMMA. *For each nice $g : \mathbb{O}'$ the comparison*

$$g^\alpha\zeta + \beta \ \leq\ g^{\alpha+\beta}\zeta$$

holds for all ordinals α, β, ζ with ζ nice.

Proof. After fixing α, ζ this follows by induction on β.

The base case $\beta = 0$ is immediate.

For the induction step, $\beta \mapsto \beta + 1$, the induction hypothesis gives

$$g^{\alpha+\beta}\zeta \geq g^\alpha\zeta \geq \zeta > 0$$

(where the central comparison follows since either $\alpha = 0$ and $g^\alpha\zeta = \zeta$, or $\alpha \neq 0$ and then g^α is nice). Thus, using the inflationary property of g, we have

$$g^{\alpha+\beta+1}\zeta \ =\ g(g^{\alpha+\beta}\zeta)\ >\ g^{\alpha+\beta}\zeta\ \geq\ g^\alpha\zeta + \beta$$

to give the required result.

For the induction leap to a limit ordinal λ we have

$$\begin{aligned} g^{\alpha+\lambda}\zeta &= \bigvee\{g^\gamma\zeta \mid \gamma < \alpha + \lambda\} \\ &\geq \bigvee\{g^{\alpha+\beta}\zeta \mid \beta < \lambda\} \\ &\geq \bigvee\{g^\alpha\zeta + \beta \mid \beta < \lambda\} \\ &= g^\alpha\zeta + \bigvee\{\beta \mid \beta < \lambda\} \ =\ g^\alpha\zeta + \lambda \end{aligned}$$

as required. ■

We use two particular cases of this comparison.

5 CORRECTION. *For each nice* $g : \mathbb{O}'$ *both*

- $\alpha < \beta \Rightarrow g^\alpha\zeta < g^\beta\zeta$
- $\zeta + \beta \leq g^\beta\zeta$

hold for all ordinals α, β, ζ *with* ζ *nice.*

A little later we will need to show that certain other operators are monotone.

6 LEMMA. *For each nice* $g : \mathbb{O}'$, $\zeta : \mathbb{O}$, *if the ordinal* ν *satisfies*

$$g^\nu\zeta = \nu$$

then

$$g^\nu\eta = \nu$$

holds for all $0 < \eta \leq \zeta$.

Proof. Note that $\nu \neq 0$, otherwise $\zeta = f^0\zeta = \nu = 0$, which is excluded. The iterate f^ν is nice, hence

$$f^\nu\eta \leq f^\nu\zeta = \nu$$

which, with $\nu \leq \eta + \nu \leq f^\nu\eta$, gives the required result. ■

The equation (5) are solved in a uniform way.

7 DEFINITION. Let the operators

$$\Diamond : \mathbb{O}' \to \mathbb{O} \to \mathbb{O}' \quad , \quad \Box : \mathbb{O}''$$

be given by

$$\Diamond h\zeta\alpha = h^\alpha\zeta \quad , \quad \Box h\zeta = (\Diamond h\zeta)^\omega\zeta$$

for all $h : \mathbb{O}'$ and $\alpha, \zeta : \mathbb{O}$. ■

The operator $\Box$ extracts the canonical solution of (5).

8 THEOREM. *For each nice* $g : \mathbb{O}'$, $\zeta : \mathbb{O}$ *the ordinal*

$$\nu = \Box g\zeta$$

satisfies

$$g^\nu\zeta = \nu$$

and is the least such ordinal.

Proof. For each $r < \omega$ let

$$\nu[r] = (\Diamond g\zeta)^r\zeta.$$

Thus,

$$\nu[0] = \zeta$$

and

$$\begin{aligned} \nu[r+1] &= (\Diamond g\zeta)^{r+1}\zeta \\ &= (\Diamond g\zeta)((\Diamond g\zeta)^r\zeta) \\ &= \Diamond g\zeta\nu[r] \qquad = g^{\nu[r]}\zeta \end{aligned}$$

for each r.

Corollary 5 gives

$$\nu[r] \le g\zeta + \nu[r] \le g^{\nu[r]}\zeta = \nu[r+1]$$

so that we have

$$\zeta = \nu[0] \le \nu[1] \le \cdots \le \nu[r+1] \le \cdots$$

an ascending sequence. Note also that

$$\begin{aligned} \nu &= \Box g\zeta \\ &= (\Diamond g\zeta)^{\omega}\zeta \\ &= \bigvee\{(\Diamond g\zeta)^{r}\zeta \mid r < \omega\} \quad = \quad \bigvee\{\nu[r] \mid r < \omega\} \end{aligned}$$

and we may use $\nu[\cdot]$ as the fundamental sequence of this defined limit ordinal ν.

From this we have

$$\begin{aligned} g^{\nu}\zeta &= \bigvee\{g^{\alpha}\zeta \mid \alpha < \nu\} \\ &= \bigvee\{g^{\nu[r]}\zeta \mid r < \omega\} \\ &= \bigvee\{\nu[r+1] \mid r < \omega\} \quad = \quad \nu \end{aligned}$$

where Corollary 5 has been used.

Finally, if

$$g^{\mu}\zeta = \mu$$

then, by a simple argument, we have $\zeta \le \mu$, and $\nu[r] \le \mu$ follows by induction on r.

This completes the proof. ■

How does this operator $\Box$ help us to generate the Veblen hierarchy?

5 The Veblen jump

[`Held in 100../41../500-bit...   Last changed September 12, 1994`]

For each normal function $f : \mathbb{O}'$ and $\zeta : \mathbb{O}$ the function

$$\alpha \mapsto (\mathrm{Fix}\, f)^{\alpha}\zeta$$

provides an enumeration of the fixed points of f beyond ζ. Only $\alpha = 0$ does not give such a fixed point. Since $\mathrm{Fix} f\zeta$ is a fixed pont of f, we have $\mathrm{Fix} f\zeta = f'\overline{\zeta}$ for some $\overline{\zeta} \le \zeta$ A simple induction then shows that

$$(\mathrm{Fix}\, f)^{1+\alpha}\zeta \;=\; f'(\overline{\zeta} + \alpha)$$

for all $\alpha : \mathbb{O}$. For the case $\zeta = \omega$ our restricted version of normality gives $\overline{\zeta} = 0$.

With this we have

$$\begin{aligned} \Box(\mathrm{Fix}\, f)\zeta &= \text{least } \nu \text{ with } (\mathrm{Fix}\, f)^{\nu}\zeta = \nu \\ &= \text{least } \nu \text{ with } f'(\overline{\zeta} + \nu) = \nu \\ &= \text{least } \nu \text{ with } \zeta < \nu = f'\nu \quad = \quad \mathrm{Fix}\, f'\zeta \end{aligned}$$

for all (suitable) normal $f : \mathbb{O}'$ and nice $\zeta : \mathbb{O}$. In this calculation we have assumed that the least ν satisfying

$$f'(\overline{\zeta} + \nu) = \nu$$

is so large in comparison with $\overline{\zeta}$ that $1 + \nu = \overline{\zeta} + \nu = \nu$. This is a modest requirement on f and is true for all normal functions we want to use in practice, but just to be on the safe side I have put in a warning condition of suitability.

This result show that

$$\Box(\mathrm{Fix}\, f)\zeta \;=\; \mathrm{Fix} f'\zeta$$

holds for all (suitable) normal $f : \mathbb{O}'$ and nice ζ. Notice how this virtually uncouples the effect of the ordinal argument ζ. We extend this to show

$$\Box^{\beta}(\mathrm{Fix}\, f)\zeta \;=\; \mathrm{Fix}\, f^{(\beta)}\zeta \tag{6}$$

for all (suitable) normal $f : \mathbb{O}'$ and $\beta, \zeta : \mathbb{O}$ with ζ nice.

This is proved by induction on β, and only the induction leap to a limit ordinal μ is not immediate. For this case we have, for each $\zeta : \mathbb{O}$

$$\begin{aligned}
\Box^{\mu}(\mathrm{Fix}\, f)\zeta \;&=\; \bigvee\{\,\Box^{\beta}(\mathrm{Fix}\, f)\zeta \mid \beta < \mu\} \\
&=\; \bigvee\{\mathrm{Fix}\, f^{(\beta)}\zeta \mid \beta < \mu\} \\
&=\; \text{the least } \nu \text{ with } \zeta < \nu = f^{(\beta)}\nu \text{ for all } \beta < \mu \\
&=\; \text{the least } \nu \text{ with } \zeta < \nu = f^{(\mu)}\nu \quad = \; \mathrm{Fix}\, f^{(\mu)}\zeta
\end{aligned}$$

to give the required result. You should give some careful thought to the reasoning behind the third step.

We can now use (3,4,6) to get

$$\phi_f(1+\beta, \alpha) \;=\; f^{(\beta)\prime}\alpha \;=\; (\mathrm{Fix}\, f^{(\beta)})^{1+\alpha}\omega \;=\; (\,\Box^{\beta}(\mathrm{Fix}\, f))^{1+\alpha}\omega$$

i.e.

$$\phi_f(1+\beta, \alpha) \;=\; (\,\Box^{\beta}(\mathrm{Fix}\, f))^{1+\alpha}\omega$$

for all $\alpha, \beta : \mathbb{O}$ and (suitable) normal $f : \mathbb{O}'$. This is the rephrasing of the construction of the Veblen hierarchy.

As a particular case we have

$$f^{+}(1+\beta) \;=\; \phi_f(1+\beta, 0) \;=\; \Box^{\beta}(\mathrm{Fix}\, f)\omega$$

so we are now interested in solving

$$\Box^{\nu}(\mathrm{Fix}\, f)\zeta \;=\; \nu$$

for given $\zeta : \mathbb{O}$. We can see a theme developing here.

6 Nice operators

`[Held in 100../41../600-bit... Last changed September 12, 1994]`

What have we done so far? We have defined a notion of niceness for inhabitants of $\mathbb{O}$ and $\mathbb{O}'$, and we have shown how each normal function $f : \mathbb{O}'$ produces a nice function $\mathrm{Fix}f : \mathbb{O}'$. We have also constructed an operator $\Box : \mathbb{O}''$ which extracts the solutions of certain fixed point equations. In particular, for a given normal function $f : \mathbb{O}'$, appropriate combinations of ordinal iterates of $\Box$ and $\mathrm{Fix}f$ generate the Veblen hierarchy $\phi_f(\cdot,\cdot)$ on f.

We are now ready to start moving up the higher order types

$$\mathbb{O}'', \mathbb{O}''', \mathbb{O}'''', \ldots, \mathbb{O}^{(k+1)}, \ldots \tag{7}$$

to produce particular inhabitants.

We will define a notion of niceness for each of these types. We will also mimic the construction of $\Box$ to produce operators

$$\boxed{0} : \mathbb{O}''' \quad , \quad \boxed{1} : \mathbb{O}'''' \quad , \ldots, \quad \boxed{l} : \mathbb{O}^{(l+3)}, \quad \ldots \tag{8}$$

in such a way that each of $\Box$, $\boxed{0}$, $\boxed{1}$, ... is nice (at the appropriate level). Note how these operators are indexed. As you read this development you may think that they should be indexed in a different way. However, after experimenting with several versions I believe the present system allows the smoothest development

The operators (8) will be generated in a uniform fashion by mimicking the construction of $\Box$. Furthermore, each instance $\boxed{l}$ of this construction will be representable in $\boldsymbol{\lambda}\mathbf{2}$ `and, I believe, this will provide a collection of ordinals which cofinally exhaust those representable in` $\boldsymbol{\lambda}\mathbf{2}$.

Consider first a general type $\mathbb{O}^{(k+2)}$ in the sequence (7). Thus k is any natural number and the particular case $k = 0$ gives $\mathbb{O}''$. This type may be decomposed as

$$\mathbb{O}^{(k+2)} = \mathbb{O}^{(k)\prime} \to \mathbb{O}^{(k)} \to \cdots \to \mathbb{O}' \to \mathbb{O} \to \mathbb{O}$$

to display how an inhabitant $H : \mathbb{O}^{(k+2)}$ must receive its arguments. Thus H requires a sequence

$$h : \mathbb{O}^{(k)\prime},\ h_k : \mathbb{O}^{(k)},\ \ldots,\ h_1 : \mathbb{O}',\ \zeta : \mathbb{O}$$

of arguments to produce

$$Hhh_k \cdots h_1\zeta : \mathbb{O}.$$

an ordinal value. For the most part the parameters $h_k, \ldots, h_1$ do not have a great effect on the calculations and when $k = 0$ they are not there at all. It is therefore convenient to condense the list

$$h_k \cdots h_1 \quad \text{to} \quad \boldsymbol{h}$$

and write

$$Hh\boldsymbol{h}\alpha \quad \text{for} \quad Hhh_k \cdots h_1\alpha.$$

This condensing notation should be used with some care for it appears to conflict with the bracketing convention.

9 DEFINITION. For each $k < \omega$, an operator $G : \mathbb{O}^{(k+2)}$ is nice if it is monotone and inflationary in the sense that

(m) $\alpha \leq \beta \Rightarrow G\boldsymbol{g}g\alpha \leq G\boldsymbol{g}g\beta$
(i) $g\boldsymbol{g}\zeta < G\boldsymbol{g}g\zeta$

hold for all nice

$$g : \mathbb{O}^{(k+1)},\ g_k : \mathbb{O}^{(k)},\ \dots,\ g_1 : \mathbb{O}^{(1)}$$

and nice ordinals α, β, ζ. ■

This, of course, is a definition by recursion. Starting from the known niceness properties for $\mathbb{O}$ and $\mathbb{O}'$ we use the Definition to generate the niceness properties for the sequence (7). I will use g, G for arbitrary nice operators.

The first new niceness property is that for $\mathbb{O}''$ (in which case the list $\boldsymbol{g}$ is empty). We already have an example such an operator.

10 LEMMA. *The operator* $\Box : \mathbb{O}''$ *is nice.*

Proof. The required monotone property follows by Lemma 6.
Consider

$$\nu = \Box g\zeta$$

where $g : \mathbb{O}'$ and $\zeta : \mathbb{O}$ are given nice inhabitants. Then

$$g^{\nu}\zeta = \nu$$

(and ν is the least such ordinal). We can not have $\nu = 0$, for otherwise

$$\zeta = g^0\zeta = \nu = 0$$

which is specifically excluded. We can not have $\nu = 1$, for otherwise

$$0 < \zeta < g\zeta = \nu = 1$$

which is contradictory. Thus $1 < \nu$ and hence Corollary 5 gives

$$g\zeta < g^{\nu}\zeta = \nu\,\Box g\zeta$$

to verify the required inflationary property. ■

We now repeat and extend the development of Section 4. I will state without proof the first few properties.

- If $G : \mathbb{O}^{(k+2)}$ and $g : \mathbb{O}^{(k+1)}$ are nice then so is $Gg : \mathbb{O}^{(k+1)}$.
- The composite $G \circ H$ of two nice operators $G, H : \mathbb{O}^{(k+2)}$ is nice.
- The pointwise supremum of a non-empty collection $\mathcal{G} \subseteq \mathbb{O}^{(k+2)}$ of nice operators is again nice.

· For each nice $G : \mathbb{O}^{(k+2)}$ the non-zero ordinal iterates

$$(G^\alpha \,|\, \alpha \in \mathbb{O},\ \alpha \neq 0)$$

are all nice.

I will deal with the comparison properties in full.

11 LEMMA. *For each $l < \omega$ and nice*

$$G : \mathbb{O}^{(l+2)},\ g : \mathbb{O}^{(l+1)},\ g_l : \mathbb{O}^{(l)},\ \dots,\ g_1 : \mathbb{O}'$$

the comparison

$$G^\alpha g\boldsymbol{g}\zeta + \beta \leq G^{\alpha+\beta} g\boldsymbol{g}\zeta$$

holds for all ordinals α, β, ζ with ζ nice.

Proof. After fixing α, ζ and the parameters $g, g_k, \dots, g_1$ (where $\boldsymbol{g} = g_k \cdots g_1$) this follows by induction on β.

The base case $\beta = 0$ is immediate.

For the induction step, $\beta \mapsto \beta + 1$, the induction hypothesis gives

$$G^{\alpha+\beta} g\boldsymbol{g}\zeta \geq G^\alpha g\boldsymbol{g}\zeta \geq \zeta > 0$$

(where the central comparison follows since $G^\alpha g\boldsymbol{g}$ is nice for all α, even $\alpha = 0$). Thus, remembering that $G^{\alpha+\beta} g$ is nice, we have

$$G^{\alpha+\beta+1} g\boldsymbol{f}\zeta = G(G^{\alpha+\beta} f)\boldsymbol{f}\zeta > G^{\alpha+\beta} g\boldsymbol{g}\zeta \geq G^\alpha g\boldsymbol{g}\zeta + \beta$$

as required.

For the induction leap to a limit ordinal λ we have

$$\begin{aligned} G^{\alpha+\lambda} g\boldsymbol{g}\zeta &= \bigvee\{G^\gamma g\boldsymbol{g}\zeta \,|\, \gamma < \alpha + \lambda\} \\ &\geq \bigvee\{G^{\alpha+\beta} g\boldsymbol{g}\zeta \,|\, \beta < \lambda\} \\ &\geq \bigvee\{G^\alpha g\boldsymbol{g}\zeta + \beta \,|\, \beta < \lambda\} \\ &= G^\alpha g\boldsymbol{g}\zeta + \bigvee\{\beta \,|\, \beta < \lambda\} \ = \ G^\alpha g\boldsymbol{g}\zeta + \lambda \end{aligned}$$

as required. ■

As before we need two particular instances of this comparison.

12 COROLLARY. *For each $l < \omega$ and nice*

$$G : \mathbb{O}^{(l+2)},\ g : \mathbb{O}^{(l+1)},\ g_l : \mathbb{O}^{(l)},\ \dots,\ g_1 : \mathbb{O}'$$

both

- $\alpha < \beta \Rightarrow G^\alpha g\boldsymbol{g}\zeta < G^\beta g\boldsymbol{g}\zeta$
- $g\boldsymbol{g}\zeta + \beta \leq G^\beta g\boldsymbol{g}\zeta$

hold for all ordinals α, β, ζ with ζ nice.

A little later we will need a monotone property.

13 LEMMA. *For each $l < \omega$ and nice*

$$G : \mathbb{O}^{(l+2)},\ g : \mathbb{O}^{(l+1)},\ g_l : \mathbb{O}^{(l)},\ \dots,\ g_1 : \mathbb{O}',\ \zeta : \mathbb{O}$$

if the ordinal ν satisfies

$$G^{\nu} g\boldsymbol{g}\zeta = \nu$$

then

$$G^{\nu} g\boldsymbol{g}\eta = \nu$$

holds for all $0 < \eta \leq \zeta$.

Proof. A simple argument show that $\nu \neq 0$ and hence

$$G^{\nu} g\boldsymbol{g}\eta \leq G^{\nu} g\boldsymbol{g}\zeta = \nu$$

since $F^{\nu} g\boldsymbol{g}$ is nice. Also, using Corollary 12

$$\nu \leq g\boldsymbol{g}\eta + \nu \leq G^{\nu} g\boldsymbol{g}\eta$$

to give the required result. ■

We are now ready to construct the operators (8) and show that they are all nice. For a given $l < \omega$ let

$$\rho_l = \mathbb{O}^{(l)},\ \dots,\ \rho_1 = \mathbb{O}^{(1)}$$

and set

$$\sigma = \mathbb{O}^{(l+1)}$$

so that $\mathbb{O}^{(l+2)} = \sigma'$ and

$$\mathbb{O}^{(l+3)} = \sigma'' = \sigma' \to \sigma \to \rho_l \to \cdots \to \rho_1 \to \mathbb{O}'.$$

The case $l = 0$ is allowed, in which case the parameter types $\rho_l, \dots, \rho_1$ disappear. We use the abbreviations in the following Definition.

14 DEFINITION. Using the abbreviations described above, for each $l < \omega$ let

$$\langle\iota\rangle : \sigma' \to \sigma \to \rho_l \to \cdots \to \rho_1 \to \mathbb{O} \to \mathbb{O}' \quad , \quad [\iota] : \sigma'' = \mathbb{O}^{(l+3)}$$

be the operations given by

$$\langle\iota\rangle Hh\boldsymbol{h}\zeta\alpha = H^{\alpha}h\boldsymbol{h}\zeta \quad , \quad [\iota]Hh\boldsymbol{h}\zeta = (\langle\iota\rangle Hh\boldsymbol{h}\zeta)^{\omega}\zeta$$

for all

$$H : \sigma',\ h : \sigma,\ h_l : \rho_l,\ \dots,\ h_1 : \rho_1$$

and $\alpha, \zeta : \mathbb{O}$ (where $\boldsymbol{h}$ is the usual condensing). ■

Note that the iterate $(\cdot)^{\alpha}$ used in the construction of $\langle\iota\rangle$ is that on σ, whereas the iterate $(\cdot)^{\omega}$ used in the construction of $[\iota]$ is that on $\mathbb{O}$.

What do these operators do?

15 THEOREM. *For each $l < \omega$ and nice*

$$G : \mathbb{O}^{(l+2)},\ g : \mathbb{O}^{(l+1)},\ g_l : \mathbb{O}^{(l)},\ \dots,\ g_1 : \mathbb{O}',\ \zeta : \mathbb{O}$$

the ordinal

$$\nu = [\imath]Gg\boldsymbol{g}\zeta$$

satisfies

$$G^{\nu}g\boldsymbol{g}\zeta = \nu$$

and is the least such ordinal.

Proof. For each $r < \omega$ let

$$\nu[r] = (\langle\imath\rangle Gg\boldsymbol{g}\zeta)^{r}\zeta.$$

Thus,

$$\nu[0] = \zeta$$

and

$$\begin{aligned} \nu[r+1] &= (\langle\imath\rangle Gg\boldsymbol{g}\zeta)^{r+1}\zeta \\ &= (\langle\imath\rangle Gg\boldsymbol{g}\zeta)((\langle\imath\rangle Gg\boldsymbol{g}\zeta)^{r}\zeta) \\ &= \langle\imath\rangle Gg\boldsymbol{g}\zeta\nu[r] \qquad = G^{\nu[r]}g\boldsymbol{f}\zeta \end{aligned}$$

for each r.

Corollary 12 gives

$$\nu[r] \le g\boldsymbol{g}\zeta + \nu[r] \le G^{\nu[r]}g\boldsymbol{g}\zeta = \nu[r+1]$$

so that we have

$$\zeta = \nu[0] \le \nu[1] \le \cdots \le \nu[r+1] \le \cdots$$

an ascending sequence. Note also that

$$\begin{aligned} \nu &= [\imath]Gg\boldsymbol{f}\zeta \\ &= (\langle\imath\rangle Gg\boldsymbol{g}\zeta)^{\omega}\zeta \\ &= \bigvee\{(\langle\imath\rangle Gg\boldsymbol{g}\zeta)^{r}\zeta \mid r < \omega\} \quad = \bigvee\{\nu[r] \mid r < \omega\} \end{aligned}$$

and we may use $\nu[\cdot]$ as the fundamental sequence of this defined limit ordinal ν.

From this we have

$$\begin{aligned} G^{\nu}g\boldsymbol{g}\zeta &= \bigvee\{G^{\alpha}g\boldsymbol{g}\zeta \mid \alpha < \nu\} \\ &= \bigvee\{G^{\nu[r]}g\boldsymbol{g}\zeta \mid r < \omega\} \\ &= \bigvee\{\nu[r+1] \mid r < \omega\} \quad = \nu \end{aligned}$$

where Corollary 12 has be used.

Finally, if

$$G^{\mu}g\boldsymbol{g}\zeta = \mu$$

then, by a simple argument, we have $\zeta \le \mu$, and $\nu[r] \le \mu$ follows by induction on r.

This completes the proof. ■

With this we have a whole battery of nice operators.

16 THEOREM. *For each $l < \omega$, the operator $[\imath] : \mathbb{O}^{(l+3)}$ is nice.*

Proof. The monotone property follows from Lemma 13.

Let $\nu = [\iota]Gg\boldsymbol{g}\zeta$ for the usual $G, g, \boldsymbol{g}, \zeta$. Then

$$G^{\nu}g\boldsymbol{g}\zeta = \nu$$

and a simple argument shows that $1 < \nu$. Corollary 12 now gives

$$Gg\boldsymbol{g}\zeta < G^{\nu}g\boldsymbol{g}\zeta = \nu = [\iota]Gg\boldsymbol{g}\zeta$$

to verify the inflationary property. ■

It is time to put these operators to work.

7 The Veblen leap

`[Held in 100../41../700-bit...  Last changed September 12, 1994]`

The new operators $[0], [1], [2], \ldots$ are very powerful. Let me illustrate this.

Each normal function f generates a sequence

$$(f^{[r]} \mid r < \omega)$$

of 'symbolic powers' as follows.

$$\begin{array}{lcl} f^{[0]} & = & id_{\mathbb{O}} \\ f^{[1]} & = & \mathrm{Fix} f \\ f^{[2]} & = & \Box(\mathrm{Fix} f) \\ f^{[3]} & = & [0]\,\Box(\mathrm{Fix} f) \\ & \vdots & \\ f^{[r+3]} & = & [r] \cdots [0]\,\Box(\mathrm{Fix} f) \\ & \vdots & \end{array}$$

Using these set

$$\Delta[r] = f^{[r]}\omega$$

for each $r < \omega$ to produce ordinals

$$\Delta[0] < \Delta[1] < \Delta[2] < \cdots$$

which, for the case $f = \omega^{\bullet}$, `I believe will cofinally exhaust the ordinals representable in` $\boldsymbol{\lambda}$`2`. To help our understanding of this we construct a second sequence

$$(f_{[r]} \mid r < \omega)$$

of normal function by

$$\begin{array}{lcl} f_{[0]} & = & f \\ f_{[1]}\alpha & = & (\mathrm{Fix} f)^{\alpha}\omega \\ f_{[2]}\alpha & = & \Box^{\alpha}(\mathrm{Fix} f)\omega \\ f_{[3]}\alpha & = & [0]^{\alpha}\,\Box(\mathrm{Fix} f)\omega \\ & \vdots & \\ f_{[r+4]}\alpha & = & [r']^{\alpha}\,[r] \cdots [0]\,\Box(\mathrm{Fix} f)\omega \\ & \vdots & \end{array}$$

where in the general case, $f^{[r+4]}$, I have written r' for $r+1$ as the index of the outermost $[\cdot]$.

Notice that

$$f_{[1]}(1+\alpha) = f'\alpha \quad , \quad f_{[2]}\alpha = f^{+}(1+\alpha)$$

and each function $f_{[r+1]}$ enumerates a set of fixed points of f which become more and more sparse as r increases. Trivially we have

$$\Delta[r] = f^{[r]}1 = f_{[r+1]}0$$

(with a slight discrepancy in the first identity for the case $r = 0$). We also have

$$\Delta[r+1] = f^{\omega}_{[r]}\omega = \mathrm{Fix} f_{[r]}\omega$$

(and I suggest you go through a proof of this).

For the particular case $f = \omega^{\bullet}$ these give the following.

$$\begin{array}{lclcl}
\Delta[0] & & & = & \omega \\
\Delta[1] & = & \mathrm{Fix} f\omega & = & \epsilon_0 \\
\Delta[2] & = & \mathrm{Fix} f'\omega & = & \epsilon_{\epsilon_{\epsilon_{\cdots}}} \\
\Delta[3] & = & \mathrm{Fix} f^{+}\omega & = & \Gamma_0
\end{array}$$

To continue further let us write

$$[{\scriptstyle 0}]\,\Box(\mathrm{Fix} f) = \mathrm{Fix}\, f^{@}$$

where $f^{@}$ is some function which dominates at least f^{+}. We may repeat the previous steps starting from $f^{@}$ in place of f to obtain some ordinal

$$\Delta^{@}[3] = [{\scriptstyle 0}]\,\Box(\mathrm{Fix} f^{@})\omega$$

which will be very much larger than Γ_0. This is still nowhere near $\Delta[4]$.

We have

$$\begin{array}{rcl}
[{\scriptstyle 0}]^2\,\Box(\mathrm{Fix} f)\omega & = & [{\scriptstyle 0}](\,[{\scriptstyle 0}]\,\Box)(\mathrm{Fix} f)\omega \\
 & & \\
 & = & \text{least } \nu \text{ with} \\
 & & (\,[{\scriptstyle 0}]\,\Box)^{\nu}(\mathrm{Fix} f)\omega \quad > \quad (\,[{\scriptstyle 0}]\,\Box)^{\gamma}(\mathrm{Fix} f)\omega
\end{array}$$

for all small $\gamma : \mathbb{O}$. The case $\gamma = 1$ gives

$$[{\scriptstyle 0}]^2\,\Box(\mathrm{Fix} f)\omega > \Delta[3]$$

the case $\gamma = 2$ gives

$$\begin{array}{rcll}
[{\scriptstyle 0}]^2\,\Box(\mathrm{Fix} f)\omega & > & (\,[{\scriptstyle 0}]\,\Box)^2(\mathrm{Fix} f)\omega & \\
 & = & (\,[{\scriptstyle 0}]\,\Box)((\,[{\scriptstyle 0}]\,\Box)(\mathrm{Fix} f))\omega & \\
 & = & (\,[{\scriptstyle 0}]\,\Box)(\mathrm{Fix} f^{@})\omega & = \quad \Delta^{@}[3]
\end{array}$$

and the cases $\gamma = 3, 4, 5, \ldots$ give larger and large ordinals. Of course the actual value of γ that we need is enormous. Now remember that

$$\Delta[4] = [{\scriptstyle 1}]\,[{\scriptstyle 0}]\,\Box(\mathrm{Fix} f)\omega = (\text{least } \nu \text{ with } [{\scriptstyle 0}]^{\nu}\,\Box(\mathrm{Fix} f)\omega = \nu)$$

and you begin to see just how big $\Delta[4]$ is.

8 Representation in $\boldsymbol{\lambda}\mathbf{2}$

`[Held in 100../41../800-bit...  Last changed September 12, 1994]`

The gadgets Fix, $\square$, $\boxed{0}$, $\boxed{1}$, ... and ordinals $\Delta[0], \Delta[1], \Delta[2], \ldots$ have been constructed in such a way that the representation in $\boldsymbol{\lambda}\mathbf{2}$ is almost routine. However, there are still one or two minor points to be considered.

Let me remind you of the essentials of $\boldsymbol{\lambda}\mathbf{2}$.

The types $\rho, \sigma, \tau, \ldots$ are generated from a stock of variables X by

$$\tau ::= X \,|\, \sigma \to \rho \,|\, \forall X.\rho$$

where the first two clauses generate the simple types, and the third clause is used to produce polymorphic types. The raw terms $r, s, t, \ldots$ of $\boldsymbol{\lambda}\mathbf{2}$ are generated from a stock of identifiers x by

$$t ::= x \,|\, \lambda x : \sigma.r \,|\, ts \,|\, \Lambda X.r \,|\, t\sigma$$

where the first three clauses generate the simple terms.

A statement is a pair

$$t : \tau$$

which is read as 'the terms t is correctly formed to inhabit τ'. There is a derivation systems which manipulates judgements

$$\Gamma \vdash t : \tau$$

where the context Γ is a finites sequence of declarations

$$x : \sigma$$

controlling the range of variation of free identifiers x of the subject t of the statement $t : \tau$. In other words, a judgment is a statement in context. (Strictly speaking, the context Γ should also list the type variables which are allowed to occur free in the statement, but we don't need to worry about that here.)

The derivation system extracts the correctly formed judgements. Simple abstraction and application are used to convert statements

$$r : \rho \qquad , \qquad t : \sigma \to \rho \,,\; s : \sigma$$

into statements

$$(\lambda x : \sigma.r) : \sigma \to \rho \qquad , \qquad ts : \rho$$

respectively. Polymorphic abstraction and application are used to convert statements

$$r : \rho \qquad , \qquad t : \forall X.\rho$$

into statements

$$(\Lambda X.r) : (\forall.\rho) \qquad , \qquad t\sigma : \rho[X := \sigma]$$

respectively. Here $\rho[X := \sigma]$ is the result of replacing the free occurrences of X in ρ by σ.

There is a reduction relation

$$t_1 \rhd\!\!\rhd t_2$$

on terms. This removes simple and polymorphic redexes from terms by

$$(\lambda x : \sigma.r)s \rhd\!\rhd r[x := s] \quad , \quad (\Lambda X.r)\sigma \rhd\!\rhd r[X := \sigma]$$

where $[x := s]$ and $[X := \sigma]$ are substitution operators.

We want to produce types and terms which represent the ordinal gadgets constructed earlier. For this we first need a representation of the natural numbers in $\boldsymbol{\lambda 2}$.

A successor structure

$$\mathfrak{S} = (\mathbb{S}, a, suc)$$

is a domain $\mathbb{S}$ furnished with a particular element $a : \mathbb{S}$ and an operation $suc : \mathbb{S}'$. By iterating suc we produce a sequence

$$a = suc^0 a,\ suc\, a,\ suc^2 s,\ suc^3 a,\ \dots$$

of elements of $\mathbb{S}$. We regard the natural number $n \in \mathbb{N}$ as a polymorphic gadget which, when supplied with a successor structure $\mathfrak{S}$ will return the element $suc^n a$ of the carrier $\mathbb{S}$ of $\mathfrak{S}$. Thus n is a kind of choice function which when supplied with an index $\mathfrak{S}$ will return a selected element from the domain indexed by $\mathfrak{S}$.

To implement this idea in $\boldsymbol{\lambda 2}$ let

$$\mathcal{N} ::= (\forall X)[X \to X' \to X]$$

and set

$$\underline{n} ::= \Lambda X \,.\, \lambda x : X, \lambda y : X' \,.\, y^n x$$

for each $n \in \mathbb{N}$. Here the compound term

$$y^n x$$

is constructed recursively by

$$\begin{aligned} y^0 x &\equiv x \\ y^1 x &\equiv yx \\ y^2 x &\equiv y(yx) \\ y^3 x &\equiv y(y^2 x) \\ &\vdots \\ y^{k+1} x &\equiv y(y^k)x \\ &\vdots \end{aligned}$$

(and so this notation conflicts with the usual bracketing convention for terms). With this we see that $\mathcal{N}$ is a type and

$$\vdash \underline{n} : \mathcal{N}$$

(and, in fact, these terms $\underline{n}$ are all the normal inhabitants of $\mathcal{N}$). Notice that when $\underline{n}$ is supplied with a carrying type σ, an inhabitant $a : \sigma$, and an inhabitant $s : \sigma'$, the term $\underline{n}\sigma a s$ reduces to the composite $s^n a$.

It is straight forward to provide representations of the standard number theoretic functions.

You may be more familiar with the type $\forall X.X''$ and the term

$$\overline{n} ::= \Lambda X \,.\, \lambda y : X', x : X \,.\, y^n x$$

as a representation of n. This version has a slightly flaw which can lead to incorrect representations of some standard functions. The type $\mathcal{N}$ used here avoids these problems.

We use the same idea to simulate various aspects of the ordinals.

A limit structure

$$\mathfrak{S} = (\mathbb{S}, a, suc, lim)$$

is a successor structure $(\mathbb{S}, a, suc)$ furnished with an extra attribute

$$lim : (\mathbb{N} \to \mathbb{S}) \to \mathbb{S}$$

which converts sequences $p : \mathbb{N} \to \mathbb{S}$ of elements into a 'limit' element $lim\, p$. Given a notation for an ordinal α (expressed in terms of the successor function on ordinals and limits of fundamental sequences) we may use this notation to produce an element $\alpha\mathfrak{S}$ of the carrier $\mathbb{S}$ of the limit structure $\mathfrak{S}$. We view the notation as a recipe for combining the attributes a, suc, lim of $\mathfrak{S}$. Note that the value $\alpha\mathfrak{S}$ depends crucially on the *notation* for α, not just on the ordinal itself.

To implement this idea in $\boldsymbol{\lambda 2}$ let

$$\mathcal{L}(X) = (\mathcal{N} \to X) \to X$$

for each variable X. More generally, for a type σ let

$$\mathcal{L}(\sigma) = (\mathcal{N} \to \sigma) \to \sigma$$

to obtain the type of 'limit creators' over σ. Set

$$\mathcal{O} ::= (\forall X)[X \to X' \to \mathcal{L}(X) \to X]$$

to obtain our polymorphic type of ordinal notations. The idea is clear. We wish to attach to as many ordinals α as possible a term $\ulcorner\alpha\urcorner$ with $\vdash \ulcorner\alpha\urcorner : \mathcal{O}$ such that when supplied with a name (σ, a, s, l) for the limit structure $\mathfrak{S}$, i.e. when supplied with

- a type σ naming a domain $\mathbb{S}$
- a term a naming an element of $\mathbb{S}$
- a term s naming an operation on $\mathbb{S}$
- a term l naming a limit creator for $\mathbb{S}$

the term

$$\ulcorner\alpha\urcorner\sigma a s l$$

will provide a recipe for naming the element $\alpha\mathfrak{S}$ of $\mathbb{S}$.

Finite iterations are easy to deal with. For each $n \in \mathbb{N}$ let

$$\ulcorner n\urcorner ::= \Lambda X\,.\,\lambda x : X, y : X', l : \mathcal{L}(X)\,.\,y^n x$$

to obtain $\vdash \ulcorner n\urcorner : \mathcal{O}$. The prefix

$$\Lambda X\,.\,\lambda x : X, y : X', l : \mathcal{L}(X)$$

will occur quite a lot (since it is the prefix of all normal inhabitants of $\mathcal{O}$). I will condense this to

$$\Lambda Xxyl$$

to make various terms easier to read. For instance

$$\ulcorner n \urcorner ::= \Lambda Xxyl \,.\, y^n x$$

and the term

$$\mathsf{fin} ::= \lambda u : \mathcal{N} \,.\, \Lambda Xxyl \,.\, uXxy$$

satisfies

$$\vdash \mathsf{fin} : \mathcal{N} \to \mathcal{O} \quad \text{and} \quad \mathsf{fin}\,\underline{n} \rhd\!\rhd \ulcorner n \urcorner$$

for all $n \in \mathbb{N}$.

The type $\mathcal{O}$ carries its own limit structure. Set

$$\mathsf{0} ::= \ulcorner 0 \urcorner = \Lambda Xxyl \,.\, x \quad , \quad \mathsf{S} = \lambda \alpha : \mathcal{O} \,.\, \Lambda Xxyl \,.\, y(\alpha Xxyl)$$

to produce a successor structure on $\mathcal{O}$. Notice how S successfully captures the step $f^\alpha \mapsto f^{\alpha+1} = f \circ f^\alpha$ for iterating functions.

Let $\mathcal{S} ::= \mathcal{N} \to \mathcal{O}$, so that $\mathcal{S}$ houses the names of fundamental sequences of limit ordinals (and other sequences of ordinals as well). Using this set

$$\mathsf{L} ::= \lambda p : \mathcal{S} \,.\, \Lambda Xxyl \,.\, l(\lambda u : \mathcal{N} \,.\, puXxyl)$$

and observe how this captures the limit leap for function iteration.

What about the ordinal ω? We view this as an instruction to iterate a supplied function all the way up the natural numbers and then take the limit using a supplied limit creator. The term

$$\omega ::= \Lambda Xxyl \,.\, l(\lambda u : \mathcal{N} \,.\, uXxy)$$

captures this idea. It is easy to check that

$$\vdash \omega : \mathcal{O} \quad , \quad \mathsf{L}\,\mathsf{fin} \rhd\!\rhd \omega$$

since, of course, $\mathsf{fin} : \mathcal{S}$ names the canonical fundamental sequence of ω.

We could now produce representations of the standard ordinal functions. It isn't necessary to do this here since we don't need the details of these representations. We can begin to look at representations of Fix, $\square$, $[0]$, ...

Consider the term

$$\mathsf{Fix} ::= \lambda h : \mathcal{O}' \,.\, \lambda \zeta : \mathcal{O} \,.\, \omega \mathcal{O}(\mathsf{I}\zeta)h\mathsf{L}$$

where $\mathsf{I} : \mathcal{O}'$ is a representation of some suitable inflationary function $\zeta \mapsto \zeta^{\cdot}$. To see this is a correct representation of Fix suppose the term $\mathsf{f} : \mathcal{O}'$ names a normal function f and $\ulcorner \zeta \urcorner : \mathcal{O}'$ names $\zeta : \mathbb{O}$. Then setting $\ulcorner \zeta^{\cdot} \urcorner = \mathsf{I}\ulcorner \zeta \urcorner$, we have

$$\mathsf{Fix}\,\mathsf{f}\ulcorner \zeta \urcorner \rhd\!\rhd \omega \mathcal{O}\ulcorner \zeta^{\cdot} \urcorner \mathsf{f}\mathsf{L} \rhd\!\rhd \mathsf{L}(\lambda u : \mathcal{N} \,.\, u\mathcal{O}\ulcorner \zeta^{\cdot} \urcorner \mathsf{f})$$

which names the limit of the sequence

$$r \mapsto f^r \zeta$$

i.e. $f^{\omega}\zeta$, which is Fix$f\zeta$.

The two operations $\Diamond$, $\Box$ (given in Definition 7) are easy to represent.

$$\Diamond ::= \lambda h : \mathcal{O}', \lambda\zeta, \alpha : \mathcal{O} . \alpha\mathcal{O}\zeta h\mathsf{L} \quad , \quad \Box ::= \lambda h : \mathcal{O}', \lambda\zeta\mathcal{O} . \omega\mathcal{O}\zeta(\Diamond h\zeta)\mathsf{L}$$

It is easy to check that

$$\vdash \Diamond : \mathcal{O}' \to \mathcal{O} \to \mathcal{O}' \quad , \quad \vdash \Box : \mathcal{O}''$$

and a few simple calculations show that the term behave in the correct way.

The representation of the operators $\langle\imath\rangle$ and $[\imath]$ is not quite so straight forward. Consider the Definition 14 of $\langle\imath\rangle$ and the abbreviations given just before. With $\sigma = \mathcal{O}^{(l+1)}$ the construction of $\langle\imath\rangle$ involves an ordinal iterate H^{α} of an operation $H : \sigma'$ and at limit stages μ this involves a pointwise supremum

$$\bigvee_{\sigma}\{H^{\alpha}h \mid \alpha < \mu\}$$

in σ. This will be computed as

$$\bigvee_{\sigma}\{H^{\mu[r]}h \mid ra < \omega\}$$

where $\mu[\cdot]$ is a chosen fundamental sequence of μ. To implement this use of $\bigvee_{\sigma}$ we need a limit creator on σ. This will be lifted from the limit creator L on $\mathcal{O}$.

Consider the following term.

$$\Uparrow ::= \Lambda X . \lambda l : \mathcal{L}(X), \lambda p : \mathcal{N} \to X', \lambda x : X . l(\lambda u : \mathcal{N} . pux)$$

It is easy to check that

$$\vdash \Uparrow : (\forall X)[\mathcal{L}(X) \to \mathcal{L}(X')]$$

and, on inspection, we see that for each type ρ, the term $\Uparrow\rho$ captures the way we lift a supremum operation on ρ (for ω-sequences) to a pointwise supremum on ρ'.

Let $(\mathsf{L}_l \mid l < \omega)$ be the sequence of terms defined by

$$\mathsf{L}_0 = \Uparrow \mathcal{O}\mathsf{L} \quad , \quad \mathsf{L}_{l+1} = \Uparrow \mathcal{O}^{(l+1)}\mathsf{L}_l$$

(for each $l < \omega$). It is easy to check that

$$\vdash \mathsf{L}_l : \mathcal{L}(\mathcal{O}^{(l+1)})$$

holds, and the two reductions

$$\begin{array}{llll} \mathsf{L}_l & \rhd\!\!\rhd & \lambda p : \mathcal{N} \to \mathcal{O}^{(l+1)}, & \rhd\!\!\rhd \ \lambda p : \mathcal{N} \to \mathcal{O}^{(l+1)}, \\ & & \quad \lambda h_l : \mathcal{O}^{(l)}, \ldots, h_1 : \mathcal{O}^{(1)}, & \quad \lambda h_l : \mathcal{O}^{(l)}, \ldots, h_1 : \mathcal{O}^{(1)}, \\ & & \quad\quad \lambda\zeta : \mathcal{O} . & \quad\quad \lambda\zeta : \mathcal{O}, \\ & & \quad\quad\quad \mathsf{L}(\lambda u : \mathcal{N} . puh_l \cdots h_1\zeta) & \quad\quad\quad \mathbf{\Lambda} Xxyl, \\ & & & \quad\quad\quad\quad \lambda u : \mathcal{N} . \\ & & & \quad\quad\quad\quad\quad pu\boldsymbol{h}\zeta Xxyl \end{array}$$

follow by a straight forward induction over l.

We now set

$$\langle\iota\rangle ::= \lambda H : \mathcal{O}^{(l+2)}, \lambda h : \mathcal{O}^{(l+1)}, \lambda h_l : \mathcal{O}^{(l)}, \dots, h_1 : \mathcal{O}^{(1)}, \lambda\zeta, \alpha : \mathcal{O} . (\alpha\mathcal{O}^{(l+1)}hH\mathsf{L}_l)\boldsymbol{h}\zeta$$

$$[\iota] ::= \lambda H : \mathcal{O}^{(l+2)}, \lambda h : \mathcal{O}^{(l+1)}, \lambda h_l : \mathcal{O}^{(l)}, \dots, h_1 : \mathcal{O}^{(1)}, \lambda\zeta : \mathcal{O} . \omega\mathcal{O}\zeta(\langle\iota\rangle Hh\boldsymbol{h}\zeta)\mathsf{L}$$

where I have used a condensing notation ($\boldsymbol{h}$ for $h_l \cdots h_1$) and I have inserted a pair of brackets in the terms for $\langle\iota\rangle$ to isolate the important part.

You should convince yourself that these are correct representations of $\langle\iota\rangle$ and $[\iota]$.

With all this machinery available representations of the ordinals $\Delta[r]$ are easy to produce.

Let f be a given term representing some base function f. For convenience let

$$\mathsf{N} = \mathsf{Fixf}$$

and think of this as the 'next fixed point' function.

In the first instance we can formally mimic the construction of the ordinals $\Delta[r]$.

$$\begin{array}{lcl} \Delta[0] & ::= & \omega \\ \Delta[1] & ::= & \mathsf{N}\omega \\ \Delta[2] & ::= & \square\mathsf{N}\omega \\ & \vdots & \\ \Delta[r+3] & ::= & [r]\cdots[0]\,\square\mathsf{N}\omega \\ & \vdots & \end{array}$$

These terms are not in normal form but can be reduced somewhat to produce rather compact (but still not normal) forms.

Consider the sequence of terms

$$\begin{array}{lcl} \delta_0 & ::= & \mathsf{f} \\ \delta_1 & ::= & \lambda\alpha : \mathcal{O} . \alpha\mathcal{O}\omega\mathsf{N}\mathsf{L} \\ \delta_2 & ::= & \lambda\alpha : \mathcal{O} . (\alpha\mathcal{O}'\mathsf{N}\,\square\,\mathsf{L}_0)\omega \\ \delta_3 & ::= & \lambda\alpha : \mathcal{O} . (\alpha\mathcal{O}''\,\square\,[0]\mathsf{L}_1)\mathsf{N}\omega \\ \delta_4 & ::= & \lambda\alpha : \mathcal{O} . (\alpha\mathcal{O}'''\,[0]\,[1]\mathsf{L}_2)\,\square\,\mathsf{N}\omega \\ & \vdots & \\ \delta_{r+5} & ::= & \lambda\alpha : \mathcal{O} . (\alpha\mathcal{O}^{(r+4)}\,[r'']\,[r']\mathsf{L}_{r+3})\,[r]\cdots[0]\mathsf{N}\omega \\ & \vdots & \end{array}$$

where, in the general case, I have written r'' for $r+2$ and r' for $r+1$. All of these represent ordinal functions and, at some stage, you might convince yourself that δ_r represents the function $f_{[r]}$ of the previous section.

Using these terms we can perform a series of reductions.

$$\begin{array}{rl} \Delta[r+1] & \rhd\!\!\rhd\ \omega\mathcal{O}\omega\delta_r\mathsf{L} \\ & \rhd\!\!\rhd\ \mathsf{L}(\lambda u : \mathcal{N} . u\mathcal{O}\omega\delta_r) \\ & \rhd\!\!\rhd\ \boldsymbol{\Lambda}Xxyl . l(\lambda u : \mathcal{N} . u\mathcal{O}\omega\delta_r Xxyl) \end{array}$$

This final term is still not in normal form because of the component δ_r, however you can see the shape of the normal form beginning to appear.

For small r the terms δ_r reduce further to obtain terms representing

$$\Delta[1] = \epsilon_0,\ \Delta[2] = \epsilon_{\epsilon_\cdot},\ \Delta[3] = \Gamma_0,\ \Delta[4] = ?,\ \ldots$$

and you may like to work out what these are.

9 Possible developments

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There are several obvious questions and directions that should be investigated.

How does this system of notations relate to other, more standard, systems? In particular, what is the relationship to the notations produced using the ϑ-functions (as described in section 7 of [29])? It seems that the ϑ-generated ordinals lie somewhere between $\Delta[3]$ and $\Delta[4]$ (unless, of course, we are allowed to use larger ordinals to index the enumeration of smaller ordinals).

How do the ordinals of proof theoretic interest fit into the system described here?

In particular, the position of Δ relative to all the ordinals representable in $\boldsymbol{\lambda}\mathbf{2}$ should be determined. Initially I thought (and claimed) that Δ was the least ordinal not representable in $\boldsymbol{\lambda}\mathbf{2}$. Peter Aczel disputed this claim and suggested that Δ bounds those ordinals representable using only a "shallow" type structure in some appropriate sense. It is thus of interest to isolate precisely what is needed to represent as a single term the construct fundamental sequence $\Delta[\cdot]$ of Δ.

Either way I suggest that Δ should occur quite naturally in various proofs of global properties of $\boldsymbol{\lambda}\mathbf{2}$.

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