About Stone's notion of Spectrum

Thierry Coquand

Introduction

The goal of this paper is to analyse two remarkable notes by Stone [StoI, StoII]. Both describe a compact space in term of some algebra of functions over this space. This description is intuitively in term of "observable" quantities. Indeed, one primary source of motivation of these notes is in operator theory, where one considers an algebra generated by elements representing observable quantities. The approach of "formal" or "point-free" topology [Men, Joh] has also the aim of describing a space not in term of "ideal" points, but in term of observable notions. We have thus two different ways of describing a space without using points, and these two ways are known classically to be equivalent, using representation theorems. A natural question arises if the formal approach can be connected to Stone's approach directly, without relying on points (or non observable notions). We present here such a connection.

This paper is organised as follows. Corresponding to the first note of Stone, we associate to a preordered ring R its real spectrum which is here defined as a distributive lattice $Spec_r(R)$ given by generators and relations. (In term of points, the points of $Spec_r(R)$ are the *prime cones* of R extending the given preorder [BCR].) If the ring satisfies some natural conditions considered in Stone's paper, we completely characterise the ordering of this lattice and we show that it is a *normal* lattice [Joh, CaC], which means in term of points that any point is contained in a unique maximal point[1]. We can then consider the maximal spectrum $Max(R)$ associated to it. There is a natural map from R to $C(Max(R))$ and we show constructively that, in a suitable sense, this map preserves the norm. This is one of the main point of Gelfand duality, which is proved non constructively in [Joh] and [BM1][2]. We show in this way that the main results in [Kri] have natural constructive proofs[3]. In particular, we obtain constructive proofs of theorems such as Kadison-Dubois [BS] by reading Krivine's arguments in a point-free setting. We then give a similar treatment for the second note of Stone. As a typical example Segal's notion of integration algebra [Seg] is expressed in our framework. We show then that in some cases, we can compute effectively the points of the maximal spectrum, like in [Bis]. Finally we explain what happens to the case of f-rings, a structure that combines the two structures considered by Stone.

Most results in this paper are elementary results about distributive lattices given by generators and relations. We think that this provides an interesting alternative approach to the spectral theorem, even in a classical framework, and we hope that this illustrates

[1] In term of lattices, it means that if $a \vee b = 1$ then we can find x, y such that $a \vee x = 1$, $b \vee y = 1$ and $x \wedge y = 0$.

[2] Our work is in the same spirit of, and directly inspired from, the work of Banaschewski and Mulvey [BM1], but carried out in a "real" framework, as opposed to the usual "complex" framework presentation of Gelfand duality.

[3] The space $Max(R)$ is called $Sp(R)$ in Krivine's paper [Kri], which does not consider the space corresponding to $Spec_r(R)$.

further the insight of Riesz [Rie] and Stone that some basic results in functional analysis can be captured by simple algebraic statements. The versions "without points" of the various representation theorems that we present imply directly their classical version as soon as we know that the spaces we consider have enough points [Joh] (in classical mathematics or in intuitionistic mathematics with some form of the fan theorem), thus providing alternative proofs of these theorems. We present some theorems in the framework of the theory of locales [Joh] but it can be worth noting that they can be formulated as well in the predicative framework of formal topology instead [Sam].

1 First representation theorem

The goal of this section is to show a representation theorem, which gives a way to represent the elements f of an ordered ring R as *continuous functions* over a compact space $Max(R)$. As we say in the introduction, this compact space is obtained from a *normal* distributive lattice $Spec_r(R)$, which is a point-free description of the usual *real spectrum* associated to R, whose points are the total preorder of R, of prime support[4]. The maximal total preorder form then a compact space $Max(R)$. This compact space $Max(R)$ will here be defined as a complete Heyting algebra given by generators and relations. The generators will be symbols $D(a)$, a in R. Each element a of R can be thought of as a continuous function $\hat{a}$ in $C(Max(R))$. One intuition is that the open $D(a)$ corresponds to the set $\{\phi \in Max(R) \mid \hat{a}(\phi) > 0\}$. It will turn out that the points of $Max(R)$ can also be thought of as ring morphisms $\sigma : R \to \mathbb{R}$ preserving positivity and that we have $\hat{a}(\sigma) = \sigma(a)$. In presence of classical logic and the axiom of choice, we recover the usual description of $Max(R)$ as a set of points. The important fact is that there are situations where one may fail to have access to the points of $Max(R)$ [BM1, MP], for instance without the axiom of choice, or working in intuitionistic logic, while our point-free description of $Max(R)$ is still possible[5].

1.1 Theory of total ordering

Let V be a preordered vector space over $\mathbb{Q}$, with a distinguished positive element 1_V. Thus V has a preorder relation $\leqslant$ (relation which is reflexive and transitive) such that $a \leqslant b$ if, and only if $a + c \leqslant b + c$. We shall use the letters $a, b, c, \ldots$ for elements of V and letters $r, s, \ldots$ for elements of $\mathbb{Q}$. We shall write 1 for 1_V, and more generally r for $r.1_V$.

Definition 1.1 $Tot(V)$ is the distributive lattice generated by the symbols $D(a)$, a in V and the axioms

$D(a) \wedge D(-a) = 0$
$D(a) = 0$ if $a \leqslant 0$
$D(a + b) \leqslant D(a) \vee D(b)$
$D(1) = 1$

Lemma 1.2 *In $Tot(V)$ we have $D(a) \leqslant D(b)$ if $a \leqslant b$. In general we have $D(a) \wedge D(b) \leqslant D(a + b)$ and $D(na) = D(a)$ if $n > 0$ and $D(s) = 1$ for $s > 0$ and $D(a - r) \vee D(s - a) = 1$ whenever $r < s$.*

[4] The *support* of a preordering is the set of elements both positive and negative. It is always an ideal.

[5] We show also later on that with a condition of separability on R and if all elements of R are *normable*, then we can also build effectively points in $Max(R)$ using only dependent choice.

Proof. If $a \leqslant b$ we have $a = b + (a - b)$ with $a - b \leqslant 0$. Hence $D(a) \leqslant D(b) \vee D(a - b)$ and $D(a - b) = 0$.

In general $a = a + b + (-b)$ and hence $D(a) \leqslant D(a + b) \vee D(-b)$. Since $0 = D(b) \wedge D(-b)$ it follows that we have $D(a) \wedge D(b) \leqslant D(a + b)$.

We have thus $D(na) \wedge D(a) \leqslant D((n + 1)a)$ and $D((n + 1)a) \leqslant D(na) \vee D(a)$ and hence $D(na) = D(a)$ for $n > 0$ by induction on n.

It follows also that we have $1 = D(r)$ for and $r > 0$ in $\mathbb{Q}$: we have $1 = D(n)$ for each natural number $n \geqslant 1$ and $1 = D(m.n/m)$ implies $1 = D(n/m)$.

Since $s - r = s - a + a - r$ it follows that $1 = D(s - a) \vee D(a - r)$ if $r < s$. $\square$

One suggestive way to read $D(a)$ is to read it as the proposition $a > 0$ for some total preordering refining the given preorder. Classically the *spectrum* of a lattice L are the lattice map $L \to \{0, 1\}$. The points of the spectrum of $Tot(V)$ can be thought of as total preordering that refines that given preorder on the vector psace V. Indeed if α is such a point and we define $a \leqslant' b$ by $\alpha(D(a - b)) = 0$ we have

$a \leqslant' a$ since $D(0) = 0$ and hence $\alpha(D(0)) = 0$
$\leqslant'$ is transitive since $D(a - c) \leqslant D(a - b) \vee D(b - c)$
$\leqslant'$ is total since $D(a - b) \wedge D(b - a) = 0$

1.2 Preordered archimedean rings

A *cone* in a ring R is a subset P which contains all squares and is closed by addition and multiplication. If P is a cone, a P*-cone* is a subset closed by addition, multiplication and containing P. The set P itself is clearly the least P-cone. If Π is a P-cone, the P-cone generated by Π and an element a in R is the set $\Pi + a\Pi$ since P and hence Π contains all the squares.

We consider a $\mathbb{Q}$-algebra R with unit 1 with a given cone P. Since P contains all squares $1/n^2.1$ it contains all elements $r.1$ with r non negative rationals. We shall write simply r instead of $r.1$. (It should be noticed however that we don't exclude the case where R is the trivial algebra $\{0\}$ and thus this notation may be ambiguous; however in practice this ambiguity is not a problem since it is always clear from the context if we mean r in $\mathbb{Q}$ or the element $r.1$ in R.) The elements of R are thought of as operators [StoI] and the elements of P are the positive operators. The relation $a \leqslant b$ defined as $b - a \in P$ is a preorder on R such that $0 \leqslant a^2$ for all a in R. The ring R is in particular a predordered vector space over the rationals, with a distinguished element 1 and we can consider the lattice $Tot(R)$ defined in the previous section.

We assume the ring R to be *archimedean* (an alternative formulation is that the ring R has a *strong unit*): for all a in R there exists r in $\mathbb{Q}$ such that $a \leqslant r$.

We will write $a \ll s$ whenever $a \leqslant s'$ for some $s' < s$ and $r \ll a$ whenever $r' \leqslant a$ for some $r' > r$. Constructively it may not be the case that the set of s such that $a \ll s$ has a greatest lower bound *sup* a in $\mathbb{R}$. If it holds we have *sup* $a < s$ in $\mathbb{R}$ iff $a \ll s$ in R. We say that a is *normable* iff the set of s such that $-s \ll a \ll s$ has a greatest lower bound $||a|| \in \mathbb{R}$.

Definition 1.3 $Spec_r(R)$ is the distributive lattice generated by the symbols $D(a)$, $a \in R$ and the axioms are the ones of $Tot(R)$ together with

$$D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$$

Using Lemma 1.2 we see that we have $1 = D(a)$ if $0 \ll a$. The main goal of the rest of this section is to show the converse: if we have $1 = D(a)$ in the lattice $Spec_r(R)$ then we have $0 \ll a$ in R.

Lemma 1.4 *The schema $D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$ is equivalent to the conjunction of $D(a) \wedge D(b) \leqslant D(ab)$ and $D(ab) \leqslant D(a) \vee D(-b)$.*

Proof. If we have $D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$ the we have $D(ab) \leq D(a) \vee D(-b)$ since $D(a) \wedge D(b) \leqslant D(a)$ and $D(-a) \wedge D(-b) \leqslant D(-b)$. We have alse $D(a) \wedge D(b) \leqslant D(ab)$.

Conversely the schema $D(ab) \leqslant D(a) \vee D(-b)$ can also be written $D(ab) \leqslant D(-a) \vee D(b)$ since $ab = ba$. It implies thus

$$D(ab) \leqslant (D(a) \vee D(-b)) \wedge (D(-a) \vee D(b)) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$$

since $D(a) \wedge D(-a) = D(b) \wedge D(-b) = 0$. Since $(-a)(-b) = ab$ the schema $D(a) \wedge D(b) \leqslant D(ab)$ implies $D(-a) \wedge D(-b) \leqslant D(ab)$ and thus

$$(D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b)) \leqslant D(ab)$$

$\square$

The points of the spectrum of $Spec_r(R)$ can be thought of as the *prime cone* that extends the given cone on R [BCR]. The lattice $Spec_r(R)$ can be thought of as a point-free description of the *real spectrum* of R [BCR].

Lemma 1.5 *In $Spec_r(R)$ we have if $s \geqslant 0$*

$$D(s^2 - a^2) = D(s - a) \wedge D(s + a) \qquad D(a^2 - s^2) = D(a - s) \vee D(-a - s)$$

Proof. We have

$$D(s^2 - a^2) = (D(s - a) \wedge D(s + a)) \vee (D(-s + a) \wedge D(-s - a))$$

Since $D(-s + a) \wedge D(-s - a) \leqslant D(-2s) = 0$ by Lemma 1.2 we get

$$D(s^2 - a^2) = D(s - a) \wedge D(s + a)$$

We have also

$$D(a^2 - s^2) = (D(a - s) \wedge D(a + s)) \vee (D(-a + s) \wedge D(-a - s))$$

Since $D(a - s) \leqslant D(a + s)$, $D(-a - s) \leqslant D(-a + s)$ by Lemma 1.2 we have $D(a - s) = D(a - s) \wedge D(a + s)$ and $D(-a - s) = D(-a + s) \wedge D(-a - s)$ and hence

$$D(a^2 - s^2) = D(a - s) \vee D(-a - s)$$

$\square$

The definition of $Spec_r(R)$ should be compared to Joyal's point-free definition of the Zariski spectrum of R [Joy], seen as a ring, which is defined as the distributive lattice generated by the symbols $I(a)$, a in R and the axioms

$I(0) = 0$
$I(a+b) \leqslant I(a) \vee I(b)$
$I(1) = 1$
$I(ab) = I(a) \wedge I(b)$

These axioms are satisfied if we interpret $I(a)$ as $D(a) \vee D(-a)$ in $Spec_r(R)$[6].

We shall need the following characterisation of $Spec_r(R)$, stated in [CC], which holds more generally for all commutative rings R with a preorder such that all square are positive, but not necessarily divisible or archimedean. This is essentially a version of the formal Positivstellensatz [BCR, CLR].

Lemma 1.6 *For all a, b, x, y in R we have*

$$D(a) \wedge D(x+ay) \leqslant D(x) \vee D(y) \qquad\qquad D(x+(-b)y) \leqslant D(x) \vee D(y) \vee D(b)$$

Proof. This follows from Lemma 1.2. □

Lemma 1.7 *We have*

$$D(a_1) \wedge \ldots \wedge D(a_k) \leqslant D(b_1) \vee \ldots \vee D(b_l)$$

in $Spec_r(R)$ if we have a relation $m + p = 0$ where m belongs to the multiplicative monoid generated by $a_1, \ldots, a_k$ and p belongs to the P-cone generated by $a_1, \ldots, a_k, -b_1, \ldots, -b_l$.

Proof. Let $a_1, \ldots, a_k, b_1, \ldots, b_l$ in R be given and let M be the multiplicative monoid generated by $a_1, \ldots, a_k$, and C_j be the P-cone generated by $a_1, \ldots, a_k, -b_1, \ldots, -b_j$ for $j \leqslant l$ and C^i be the P-cone generated by $a_1, \ldots, a_i$ for $i \leqslant k$. We have $C_0 = C^k$ and $C^0 = P$. An element of C_j is of the form $x + (-b_j)y$ with x, y in C_{j-1} and an element of C^i is of the form $x + a_i y$ with x, y in C^{i-1}.

We deduce from this that we have

$$D(a_1) \wedge \ldots \wedge D(a_k) \wedge D(-p) \leqslant D(b_1) \vee \ldots \vee D(b_l)$$

whenever p in C^i by induction on i: this holds for $i = 0$ since $D(-p) = 0$ if p is in P and if it holds for i it holds for $i+1$ using Lemma 1.6 and the fact that an element of C^{i+1} is of the form $x + a_{i+1}y$ with x, y in C^i. Similarly using $D(x + (-b)y) \leqslant D(x) \vee D(y) \vee D(b)$ (Lemma 1.6) we prove that we have

$$D(a_1) \wedge \ldots \wedge D(a_k) \wedge D(-p) \leqslant D(b_1) \vee \ldots \vee D(b_l)$$

whenever p in C_j by induction on j.

The fact that we have $D(a_1) \wedge \ldots \wedge D(a_k) \leqslant D(m)$ whenever m is in M follows from the schema $D(a) \wedge D(b) \leqslant D(ab)$ and the equality $1 = D(1)$. □

[6]In term of points this corresponds to the fact that if C is a prime cone of R, then $C \cap (-C)$ is a prime ideal of R.

This lemma gives a special way of proving en inequality

$$D(a_1) \wedge \ldots \wedge D(a_k) \wedge D(-p) \leqslant D(b_1) \vee \ldots \vee D(b_l)$$

The next fundamental Theorem states that this way of proving inequality is complete. (This is essentially a form of cut-elimination result [Sco].)

Theorem 1.8 *We have*

$$D(a_1) \wedge \ldots \wedge D(a_k) \leqslant D(b_1) \vee \ldots \vee D(b_l)$$

in $Spec_r(R)$ iff we have a relation $m + p = 0$ where m belongs to the multiplicative monoid generated by $a_1, \ldots, a_k$ and p belongs to the P-cone generated by $a_1, \ldots, a_k, -b_1, \ldots, -b_l$.

Proof. We define the relation $X \vdash Y$ between finite subsets X, Y of R by: there exists a relation $m + p$ where m is in the multiplicative monoid generated by X and p is in the P-cone generated by X and $-Y$. We show first that such a relation is an entailment relation [CC, Sco]: it satisfies the three laws

$X \vdash Y$ if X meets Y (reflexivity)
$X \vdash Y$ if $X' \vdash Y'$ and $X' \subseteq X$ and $Y' \subseteq Y$ (monotonicity)
$X \vdash Y$ if $X, x \vdash Y$ and $X \vdash Y, x$ (transitivity)

We notice then next that this relation is the *least* entailment relation satisfying the laws corresponding to the definition of $Spec_r(R)$ (using Lemma 1.4)

$a, -a \vdash \emptyset$
$a \vdash \emptyset$ if a is in $-P$
$a + b \vdash a, b$
$\emptyset \vdash 1$
$a, b \vdash ab$
$ab \vdash a, -b$

Theorem 1.8 follows then from the general theory of entailment relations and distributive lattices developped in [CC].

If we know that this relation is an entailment relation, that fact that this is the least entailment relation satisfying these laws is seen as follows. First it satsfies all these laws

$a, -a \vdash \emptyset$ since $a + (-a) = 0$
$a \vdash \emptyset$ if a is in $-P$ since $a + (-a) = 0$
$a + b \vdash a, b$ since $(a + b) + (-a) + (-b) = 0$
$\emptyset \vdash 1$ since $1 + (-1) = 0$
$a, b \vdash ab$ since $ab + (-ab) = 0$
$ab \vdash a, -b$ since $ab + (-a)b = 0$

Second, if S is a relation satisfying all these laws we have

$$a, x + ay \; S \; x, y \qquad\qquad x + (-b)y \; S \; x, y, b$$

We can then follow the proof of Lemma 1.7 and show that whenever we have a relation $m + p = 0$ where m is in the multiplicative monoid generated by X and p is in the P-cone generated by X and $-Y$ then we have $X \; S \; Y$.

The only remaining result to prove is that $\vdash$ is an entailment relation. Only the transitivity of the relation $\vdash$ is not direct. We show that if M is a multiplicative monoid, if C a P-cone such that $M \subseteq C$ and x in R is such that we have some relations

$$m_1 + u_1 + (-x)v_1 = 0 \qquad\qquad m_2 x^k + u_2 + xv_2 = 0$$

with m_1, m_2 in M and u_1, v_1, u_2, v_2 in C then we have 0 in $M + C$. We can rewrite the first relation as $m_1' = xv_1$ with $m_1' = m_1 + u_1$ is in $M + C$. We notice next that $M + C$ is closed by multiplication since $M \subseteq C$. The second relation implies then $m_2(xv_1)^k + u_2 v_1^k + xv_2 v_1^k = 0$ and hence is of the form $m_2' + xv = 0$ for some $m_2' \in M + C$ and v is in C. It follows that $m_1' m_2' + x^2 v_1 v = 0$ which shows that 0 is in $M + C$ as desired. □

Corollary 1.9 *We have $1 = D(b_1) \vee \ldots \vee D(b_l)$ iff an element of the cone generated by $-b_1, \ldots, -b_l$ is $\ll 0$. In this case, there exists $s > 0$ such that $1 = D(b_1 - s) \vee \ldots \vee D(b_l - s)$.*

Proof. To simplify we take $l = 2$ but the reasoning is uniform in l. By Theorem 1.8 $1 = D(b_1) \vee D(b_2)$ iff we have a relation $1 + p = 0$ where p is an element of the cone generated by $-b_1, -b_2$. Hence we have a relation of the form

$$1 + (-b_1)p_1 + (-b_2)p_2 + b_1 b_2 p = 0$$

with p_1, p_2, p in P. Since R is archimedian it follows that there exists $s > 0$ and s' such that $0 \ll s'$ and

$$s' = (-b_1 + s)(-p_1) + (-b_2 + s)(-p_2 + (-b_1 + s)(-p))$$

We have $D(s') = 1$ since $0 \ll s'$. Using Lemma 1.6

$$1 = D((-b_1 + s)(-p_1) + (-b_2 + s)(-p_2 + (-b_1 + s)(-p)))$$

implies $1 = D(b_1 - s) \vee D(b_2 - s)$ as desired. □

The next lemma and theorem are the key to our proof theoretic approach to Gelfand duality.

Lemma 1.10 *If $1 \leqslant ac$ and $0 \leqslant c$ then $0 \ll a$.*

Proof. See [Kri] Théorème 12. In order to be self-contained, and to show that the argument is elementary, we give a sketch of the argument. Since the ring is archimedean, we have N in $\mathbb{N}$ such that $a \leqslant N$. Since $0 \leqslant c$ and $1 \leqslant ac$ we have $1 \leqslant Nc$ and thus $1/N \leqslant c$. We have also L in $\mathbb{N}$ such that $c \leqslant L$ and we get $1/N \leqslant c \leqslant L$. If we write $b = 1 - c/L$ we have $0 \leqslant b \leqslant 1 - 1/NL$ and $1/L \leqslant a(1-b)$. By multiplying by $1 + \ldots + b^{n-1}$ we get $1/L \leqslant a(1-b^n)$. For n big enough we have $b^n \leqslant 1/2$ and hence $1/2L \leqslant a$. □

Theorem 1.11 *$D(a) = 1$ in $Spec_r(R)$ iff $0 \ll a$.*

Proof. The P-cone generated by $-a$ is $P + P(-a)$. It follows from theorem 1.8 that $D(a) = 1$ iff there exists $b, c \geqslant 0$ such that $1 + b + c(-a) = 0$, that is $ca = 1 + b$. The result follows then from lemma 1.10. □

The following lemma will be used only towards the end of the paper[7]. We say that a sequence of elements (x_n) in R is a Cauchy sequence iff for each $s > 0$ there exists N such that $-s \ll x_m - x_n \ll s$ if $n, m \geqslant N$.

Lemma 1.12 *For all $x \in P$ we can build a Cauchy sequence (x_n) of elements in P such that $x_n^2 \to x$.*

Proof. We can assume $0 \leqslant x \leqslant 1$. We define the two sequences (y_n) and (z_n) of elements in $[0,1]$ defined by $y_0 = z_0 = 0$ and

$$y_{n+1} = 1/2(1 - x + y_n^2) \qquad z_{n+1} = 1/2(1 + z_n^2)$$

The sequence z_n is in $\mathbb{Q}$. Clearly, we have $y_n \leqslant z_n$ for all n

I claim that we have for all n

$$y_n \leqslant y_{n+1} \quad z_n \leqslant z_{n+1} \quad y_{n+1} - y_n \leqslant z_{n+1} - z_n$$

This is proved by induction from the equalities

$$y_{n+1} - y_n = 1/2(y_n + y_{n-1})(y_n - y_{n-1}) \quad z_{n+1} - z_n = 1/2(z_n + z_{n-1})(z_n - z_{n-1})$$

It follows that we have

$$(1 - y_n)^2 - x = 2(y_{n+1} - y_n) \leqslant 2(z_{n+1} - z_n)$$

In order to conclude, all is left is to show that (z_n) has for limit 1. We know that $0 \leqslant z_n \leqslant z_{n+1} \leqslant 1$ and we have

$$1 - z_{n+1} = (1 - z_n)1/2(1 + z_n) \leqslant (1 - z_n)(1 - \epsilon/2)$$

if $z_n \leqslant 1 - \epsilon$. This shows that if $(1 - \epsilon/2)^N \leqslant \epsilon$ we have $1 - z_n \leqslant \epsilon$ for all $n \geqslant N$. □

1.3 The spectrum of an archimedean ring

A lattice L is *strongly normal* iff for any u, v in L there exists x, y in L such that $v \leqslant u \vee y$ and $u \leqslant v \vee x$ and $x \wedge y = 0$. A lattice L is *normal* iff whenever $u \vee v = 1$ there exists x, y such that $u \vee y = v \vee x = 1$ and $x \wedge y = 0$. We write $SN(u, v)$ if, and only if there exist x, y such that $v \leqslant u \vee y$ and $u \leqslant v \vee x$ and $x \wedge y = 0$. Hence L is strongly normal if, and only if we have $SN(u, v)$ for all u, v in L.

Lemma 1.13 *A strongly normal lattice is normal. If we have $SN(u_1, v)$ and $SN(u_2, v)$ then we have $SN(u_1 \vee u_2, v)$ and $SN(u_1 \wedge u_2, v)$. Hence L is strongly normal if and only if $SN(u, v)$ holds for u, v in a generating set of L.*

Proof. Assume that L is strongly normal and that $u \vee v = 1$. Since $SN(u, v)$ holds there exists x, y such that $v \leqslant u \vee y$ and $u \leqslant v \vee x$ and $x \wedge y = 0$. Since $v \leqslant u \vee y$ and $u \vee v = 1$ we have $u \vee y = 1$. Similarly $v \vee x = 1$. Hence L is normal.

Assume $SN(u_1, v)$ and $SN(u_2, v)$. We have $v \leqslant u_i \vee y_i$ and $u_i \leqslant v \vee x_i$ and $x_i \wedge y_i = 0$ for $i = 1, 2$. We have then $v \leqslant (u_1 \wedge u_2) \vee y_1 \vee y_2$ and $u_1 \wedge u_2 \leqslant v \vee (x_1 \wedge x_2)$ and $v \leqslant (u_1 \vee u_2) \vee (y_1 \wedge y_2)$ and $u_1 \vee u_2 \leqslant v \vee (x_1 \vee x_2)$ and $(x_1 \vee x_2) \wedge y_1 \wedge y_2 = (x_1 \wedge x_2) \wedge (y_1 \vee y_2) = 0$. □

[7]This is the usual lemma that R admits square root of positive elements if R is complete. Notice that the proof is directly constructive, and it corresponds to the usual Taylor expansion of $(1 - x)^{1/2}$.

Theorem 1.14 *The distributive lattice $Spec_r(R)$ is strongly normal. Its corresponding compact regular frame [CC] can be described as the frame $Max(R)$ generated by the symbols $D(a),\ a \in R$ and the relations defined by $Spec_r(R)$ together with the continuity axiom*

$$D(a) = \bigvee_{r>0} D(a-r)$$

We have $D(a) \leqslant D(b)$ in $Max(R)$ iff for all $r > 0$ there exists $s > 0$ such that $D(a-r) \leqslant D(b-s)$ in $Spec_r(R)$. The space defined by $Max(R)$ is compact completely regular [BM1, MP]

Proof. By Lemma 1.13 it is enough to show $SN(D(a), D(b))$ for all a, b in R since $Spec_r(R)$ is generated by the elements $D(a)$ for a in R. The relation $SN(D(a), D(b))$ is implied by $D(a) \leqslant D(b) \vee D(a-b)$ and $D(b) \leqslant D(a) \vee D(b-a)$ and $D(a-b) \wedge D(b-a) = 0$.

The lattice $Spec_r(R)$ defines a complete Heyting algebra of its ideals [Joh] $Idl(Spec_r(R))$.

We check next that all the relations defining $Max(R)$ are satistied for $D'(a) = \bigvee_{r>0} D(a-r)$ in $Idl(Spec_r(R))$

$D'(a) \wedge D'(-a) = 0$ holds since $D(a-s_1) \wedge D(-a-s_2) = 0$ for all $s_1, s_2 > 0$

$D'(a) = 0$ if $a \leqslant 0$ since then $D(a-s) = 0$ for all $s > 0$

$D'(a+b) \leqslant D'(a) \vee D'(b)$ since $D(a+b-s) \leqslant D(a-s/2) \vee D(b-s/2)$

$D'(1) = 1$ since $D(1-s) = 1$ if $0 < s < 1$

$D'(a) \wedge D'(b) \leqslant D'(ab)$ since we have $D(b-s_2) = D(s_1(b-s_2))$ and $D(b-s_2) \leqslant D(b)$ and hence $D(a-s_1) \wedge D(b-s_2) = (D(a-s_1) \wedge D(b)) \wedge D(s_1(b-s_2))$ is $\leqslant D((a-s_1)b) \wedge D(s_1(b-s_2)) \leqslant D(ab - s_1 s_2)$ for all $s_1, s_2 > 0$

$D'(ab) \leqslant D'(a) \vee D'(-b)$ since, using the fact that R is archimedian we can, for all $s > 0$, find $s_1, s_2 > 0$ such that $ab - s \leqslant (a-s_1)(b-s_2)$

It follows that we have a map $Max(R) \to Idl(Spec_r(R)),\ D(a) \longmapsto D'(a)$. In particular $D(a) \leqslant D(b)$ in $Max(R)$ implies $D'(a) \leqslant D'(b)$ in $Idl(Spec_r(R))$. □

Theorem 1.15 *The points of $Max(R)$ can be identified with ring morphims $\sigma{:}R \to \mathbb{R}$ such that $\sigma(a) \geqslant 0$ if $a \geqslant 0$.*

Proof. Using Lemma 1.2 we have $1 = D(a-r) \vee D(s-a)$ if $r < s$. Also $D(a-r) \wedge D(r-a) = 0$.

A point σ of $Max(R)$ associates a truth value to each generator $D(a)$ of $Max(R)$. We can then define a Dedekind real $\sigma(a)$ by taking $\sigma(a) \in (r, s)$ iff $D(a-r)$ and $D(s-a)$ become true under this interpretation. It is direct that $\sigma{:}R \to \mathbb{R}$ preserves addition and sends positive elements to positive reals, and Lemma 1.5 shows that it preserves squares, and hence multiplication. □

Thus this space coincides with the space considered by Stone [StoI]. Our results give a purely phenomenological description of this space. Since, classically, a compact regular frame has *enough points* [Joh] all statements about the space $Max(R)$ are directly equivalent to the usual statements with points. A simplification of the present real framework compared to the complex case, noticed also in [Joh], is that we don't need to rely on Gelfand-Mazur's theorem like in the reference [BM2].

1.4 Gelfand duality, main lemma

Theorem 1.16 *$D(a) = 1$ in $Max(R)$ iff $0 \ll a$.*

Proof. By theorem 1.14, if we have $1 = D(a)$ in $Max(R)$ then we have $1 = D(a - s)$ in $Spec_r(R)$ for some $s > 0$. The assertion follows then from theorem 1.11. $\square$

Corollary 1.17 *$1 = D(s - a) \wedge D(a + s)$ in $Max(R)$ iff we have $-s \ll a \ll s$ in R.*

This is one of the main lemma in establishing Gelfand's duality [Joh]. One can contrast our purely constructive development, based on theorem 1.16 with the treatment in [BM1], which is based on the non constructive use of Barr's theorem.

In general, to give a continuous function $f \in C(X)$ on a frame X is to give two families of elements of X U_r and V_s, indexed by rationals $r, s \in \mathbb{Q}$ and satisfying some conditions. Intuitively U_r stands for $f^{-1}(r, \infty)$ and V_s for $f^{-1}(-\infty, s)$. The conditions are

$\vee_r U_r = \vee_s V_s = 1$
$U_r = \vee_{r' > r} U_{r'}, \quad V_s = \vee_{s' < s} V_{s'}$
$1 = U_r \vee V_s$ if $r < s$
$0 = U_r \wedge V_s$ if $s \leqslant r$

These conditions hold if we take $X = Max(R)$ and $U_r = D(a - r)$ and $V_s = D(s - a)$ for a fixed $a \in R$. Hence any element $a \in R$ defines a continuous map $\hat{a} \in C(Max(R))$. If σ is a point of $Max(R)$ it follows from this definition that we have $\hat{a}(\sigma) = \sigma(a)$.

The corollary 1.17 can then be interpreted as a point-free formulation of the fact that the uniform norm of $\hat{a} \in C(Max(R))$ is exactly the norm of a in R. If we know that the space $Max(R)$ has enough points (in classical mathematics or in intuitionistic mathematics with some form of the fan theorem) this corollary implies directly the usual statement of Gelfand duality.

1.5 A generalisation

The corollary 1.17 can also be seen as a point-free formulation of the Kadison-Dubois theorem [BS]. The theorem of Kadison-Dubois is actually more general in that it does not assume that P contains all squares. In this subsection we show how to deal with this generalisation, following and simplifying slightly [Kr1].

Lemma 1.18 *For all n we can write $x^2 + 1 = P(n - x, n + x)$ where $P(X, Y)$ is a rational homogeneous polynomial with coefficients $\geqslant 0$.*

Proof. We use the change of variables $y(n + x) = n - x$. The question reduces to find k such that all coefficients of

$$(1 + y)^k (1 - 2\frac{n^2 - 1}{n^2 + 1} y + y^2)$$

are $\geqslant 0$. A small computation shows that this is the case iff $n^2 - 1 \leqslant k$. If we write

$$\Sigma a_i y^i = (1 + y)^k (1 - 2\frac{n^2 - 1}{n^2 + 1} y + y^2)$$

we can take $P(X, Y) = \Sigma a_i X^i Y^{k+2-i}$. $\square$

Notice that $P(X,Y)$ is of degree n^2+1. We don't know if this degree is optimal.

Corollary 1.19 *If R is a $\mathbb{Q}$-algebra with an archimedean order containing $\mathbb{Q}^+$, but without assuming that all squares are $\geqslant 0$ then we have $x^2+s \geqslant 0$ for all $x \in R$ and all rationals $s > 0$.*

Proof. Let s be a rational > 0. We can find l in $\mathbb{N}$ such that $1/l^2 \leqslant s$. Since R is archimedian, we can find n in $\mathbb{N}$ such that $-n \leqslant lx \leqslant n$. Using Lemma 1.18 we can write $1+(lx)^2 = P(n-lx, n+lx)$ where $P(X,Y)$ is a rational homogeneous polynomial with coefficients $\geqslant 0$. It follows that we have $0 \leqslant 1+(lx)^2$ and hence $0 \leqslant x^2+s$. □

Let R be a $\mathbb{Q}$-algebra with an archimedean order containing $\mathbb{Q}^+$, but without assuming that all squares are $\geqslant 0$. This means that we have a subset $\Omega \subseteq R$ closed under addition and multiplication, $\mathbb{Q}^+ \subseteq \Omega$ and for all $x \in R$ there exists n such that $n - x \in \Omega$. Let now Ω' be the *cone* generated by Ω that is the least subset of R closed under addition and multiplication, containing Ω and all squares $x^2,\ x \in R$.

Corollary 1.20 *If $x \in \Omega'$ and r is a rational > 0 then $x + r \in \Omega$.*

Proof. Let Π be the set of all elements $x \in R$ such that $x + s \in \Omega$ for all $s > 0$. We show that Π is a cone containing Ω and $\mathbb{Q}^+$. This would imply that $\Omega' \subseteq \Pi$.

if x, y are in Π then $x+y+s = (x+s/2)+(y+s/2)$ is in Ω for all $s > 0$ so Π is closed under addition

if x, y are in Π and $s > 0$ then, since R is archimedian, there exists a rational $r > 0$ such that $(x+r)(y+r) \leqslant xy+s$ and hence $xy+s$ belongs to Ω; this shows that Π is closed under multiplication

Π contains all square by Corollary 1.20

Π contains Ω since $x+s$ is in Ω if x is in Ω and s is a rational > 0, because then s belongs to Ω

□

We get a constructive proof of the following result, due to Krivine [Kri].

Theorem 1.21 *Let $Q(x_1,\dots,x_n)$ a rational polynomial which is > 0 on $[0,1]^n$, then we can write $Q = P(x_1,\dots,x_n,1-x_1,\dots,1-x_n)$ where P is a rational polynomial with all coefficients $\geqslant 0$.*

Proof. Let R be the $\mathbb{Q}$-algebra generated by $x_1,\dots,x_n$. We let Ω be the subset of R generated by addition, multiplication, and the elements $\mathbb{Q}^+$ and x_i and $1-x_i$. The polynomial $Q(x_1,\dots,x_n)$ can be seen as an element x_Q of the algebra R. If Ω' is the cone generated by Ω, and R is ordered by Ω' a point of $Max(R)$ is a ring morphism $\sigma : R \to \mathbb{R}$ such that $\sigma(x) \geqslant 0$ if x is in Ω'. This ring morphism is determined by $(\sigma(x_1),\dots,\sigma(x_n))$ in $[0,1]^n$. The hypothesis is that we have $\sigma(x) > 0$ for all σ in $Max(R)$. It can be formulated in a point-free way as the fact that we have $D(x_Q) = 1$ in $Max(R)$. The result follows then from Theorem 1.16 and Corollary 1.20: there exists $s > 0$ such that $x_Q - s$ belongs to Ω' and hence (Corollary 1.20) x_Q belongs to Ω. □

The polynomial P can be computed from any given proof that $D(x_Q) = 1$. Notice that such a proof can be computed uniformely from Q and an explicit lower bound > 0 of Q on $[0,1]^n$ by computing a finite decomposition of $[0,1]^n$ such that the variation of Q is small enough on each part.

1.6 Example

Let B be a Boolean algebra. We let R to be the $\mathbb{Q}$-algebra generated by symbols $v(b),\ b \in B$ with the relations

$$v(b_1 b_2) = v(b_1)v(b_2) \qquad v(b_1) + v(b_2) = v(b_1 b_2) + v(b_1 \vee b_2) \qquad v(1) = 1 \qquad v(0) = 0$$

We can define $0 \leqslant a$ to mean that we can write $a = \Sigma r_i v(b_i)$ with $0 \leqslant r_i$. Notice that any element $a \in R$ can be written $\Sigma r_i v(b_i)$ with $b_i b_j = 0$ if $i \neq j$. It follows from this remark that we have $0 \leqslant a^2$ for all a : indeed we have $a^2 = \Sigma r_i^2 v(b_i)$. It is clear also that we have $0 \leqslant v(b) \leqslant 1$ for all $b \in B$ and hence that R is archimedean.

Theorem 1.22 *The space $Max(R)$ is the Stone dual space of B.*

Proof. In this case $Max(R)$ coincides with the spectral frame defined by $Spec_r(R)$ and $Spec_r(R)$ coincides with B. □

The construction of this ring R is implicit in [Tar], and is useful for analysing measures on B. This is because $v{:}B \to R$ is the *universal valuation.* If $w{:}B \to S$ is another valuation in an ordered $\mathbb{Q}$-vector space S, with a distinguished positive element 1, then there exists one and only one map $f{:}R \to S$ such that $f \circ v = w$.

2 Second representation theorem

2.1 Lattice-ordered groups

Let R be now a lattice ordered abelian group (or *l-group*) [Bou, Lux]. The elements of R are written $a, b, c, \ldots$ The group operation of R is written additively and the sup operation is written $a \vee b$. We shall use the following elementary facts, that are proved in the references [Bou, Lux].

Lemma 2.1 *We have $c + (a \vee b) = (c + a) \vee (c + b)$. Any two elements a, b have an inf $a \wedge b$ and $a + b = a \wedge b + a \vee b$. If $0 \leqslant y$ and $x \perp z$ and $x \leqslant y + z$ then $x \leqslant y$. Considered as a lattice, R is distributive. For $n \geqslant 1$ we have $n(a \vee b) = na \vee nb$ and $n(a \wedge b) = na \wedge nb$, also $na \geqslant 0$ implies $a \geqslant 0$.*

This implies that R can be embedded as an l-group in a *divisible* lattice ordered group where for each x and $n \geqslant 1$ there exists exactly one solution for $ny = x$. To simplify the presentation, we will assume in the following that R is divisible; it has then naturally the structure of a Riesz space over the set of rationals $\mathbb{Q}$ [Lux].

We write as usual a^+ for $a \vee 0$ and a^- for $(-a) \vee 0$. We say that a is *positive* iff $a \geqslant 0$. Let P be the set of positive elements. We write $x \perp y$ if $x \wedge y = 0$ (notice that this implies that x and y are in P).

Lemma 2.2 *We have $a = a^+ - a^-$ and $a^+ \perp a^-$. Also if $a = b - c$ and $b \perp c$ then $b = a^+$ and $c = a^-$.*

Proof. See [Bou, Lux]. $\square$

If $b \in P$ we write $a \preceq b$ to mean that there exists $n \geqslant 1$ such that $a \leqslant nb$. If $a, b \in P$ we write $a \sim b$ iff $a \preceq b$ and $b \preceq a$.

We assume now that R has a strong unit 1: we have $0 \leqslant 1$ and $a \preceq 1$ for all $a \in R$. An important consequence is the following fact.

Lemma 2.3 *If $a \perp 1$ then $a = 0$. If $0 \ll a^+$ then $a = a^+$.*

Proof. We have $a \leqslant n1$ for some n and $a \perp 1$ implies $a \perp n1$, hence $a = 0$.

If $0 \ll a^+$ that is $1 \leqslant na^+$ for some $n > 0$ we get $a^- \perp 1$ since $a^- \perp na^+$ and hence $a^- = 0$. $\square$

The following remark will be important, and it has a direct proof from lemmas 2.1 and 2.2.

Proposition 2.4 *For $a, b, c \in P$ we have*

$$a \preceq c, b \preceq c \rightarrow a \vee b \preceq c \qquad c \preceq a, c \preceq b \rightarrow c \preceq a \wedge b,$$

hence the structure $(P/\sim, \wedge, \vee, \preceq)$ forms a distributive lattice L.

2.2 Real spectrum of an l-group

We associate to R a distributive lattice $Spec_r(R)$. It is generated by the symbols $D(a),\ a \in R$ and the axioms are the ones of the lattice $Tot(R)$ together with the schema

$$D(a \vee b) = D(a) \vee D(b)$$

Proposition 2.5 *In $Spec_r(R)$ we have $D(a^+) = D(a)$ and $D(a \wedge b) = D(a) \wedge D(b)$*

Proof. Since $D(a \vee 0) = D(a) \vee D(0)$ we get $D(a \vee 0) = D(a)$, that is $D(a^+) = D(a)$. It follows that we have $D(u^-) = D((-u)^+) = D(-u)$ and hence $D(u^+) \wedge D(u^-) = D(u) \wedge D(-u) = 0$ for all $u \in R$. Since $a = a \wedge b + (a - b)^+$ and $b = a \wedge b + (a - b)^-$ we have also

$$D(a) \leqslant D(a \wedge b) \vee D((a - b)^+) \qquad D(b) \leqslant D(a \wedge b) \vee D((a - b)^-)$$

and hence $D(a) \wedge D(b) \leqslant D(a \wedge b)$. $\square$

A similar reasoning would show.

Proposition 2.6 *In the theory describing the lattice $Tot(R)$ we have equivalence between the three schemas*

$D(a \vee b) = D(a) \vee D(b)$ for all $a, b \in R$
$D(a^+) = D(a)$ for all $a \in R$
$D(a \wedge b) = D(a) \wedge D(b)$ for all $a, b \in R$

Theorem 2.7 *The lattice $Spec_r(R)$ coincides with the lattice L with the interpretation $D(a) = a^+$. In particular, $D(a) \leqslant D(b)$ in $Spec_r(R)$ iff $a^+ \preceq b^+$.*

Proof. Using Lemma 1.2 and Proposition 2.5, we have that $a^+ \leqslant nb^+$ implies $D(a) \leqslant D(b)$ and hence if $(a_1 \wedge \ldots \wedge a_k)^+ = a_1^+ \wedge \ldots \wedge a_k^+ \preceq b_1^+ \vee \ldots \vee b_l^+ = (b_1 \vee \ldots \vee b_l)^+$ then

$$D(a_1) \wedge \ldots \wedge D(a_k) \leqslant D(b_1) \vee \ldots \vee D(b_l)$$

in $Spec_r(R)$. The other direction follows from the fact that $a \longmapsto a^+$ satisfies the conditions of $Tot(R)$ and the equality $(a \vee b)^+ = a^+ \vee b^+$. □

Corollary 2.8 *We have $1 = D(b_1) \vee \ldots \vee D(b_m)$ in $Spec_r(R)$ iff $0 \ll b_1^+ \vee \ldots \vee b_m^+$. If this holds there exists $r > 0$ such that $1 = D(b_1 - r) \vee \ldots \vee D(b_m - r)$.*

Corollary 2.9 *We have $D(a) = 1$ in $Spec_r(R)$ iff $0 \ll a$.*

Proof. If $D(a) = 1$ in $Spec_r(R)$ we have first $0 \ll a^+$ by the theorem 2.7 and then $0 \ll a$ by lemma 2.3. □

2.3 The spectrum of an archimedean divisible l-group

Theorem 2.10 *The lattice $Spec_r(R)$ is strongly normal. The corresponding compact regular frame $Max(R)$ of its maximal ideals is defined by generators $D(a)$, $a \in R$, the axioms of $Spec_r(R)$ and the continuity axiom*

$$D(a) = \bigvee_{r>0} D(a - r)$$

The frame $Max(R)$ is completely regular and its points can be identified with l-group morphims $\sigma{:}R \to \mathbb{R}$ such that $\sigma(1) = 1$.

Proof. Like for the proof of Theorem 1.14 we define an interpretation of $Max(R)$ in $Idl(Spec_r(R))$ by interpreting $D(a)$ by $D'(a) = \bigvee_{r>0} D(a - r)$ in $Idl(Spec_r(R))$. The equality $D'(a \vee b) = D'(a) \vee D'(b)$ follows from the equality $(a \vee b) - s = (a - s) \vee (b - s)$ in R. □

Corollary 2.11 *We have $D(a) = 1$ in $Max(R)$ iff $0 \ll a$.*

3 Stone-Weierstrass Theorem

Let X be an arbitrary compact completely regular locale. We let V be a sub $\mathbb{Q}$-vector space of $C(X)$ such that such $1 \in V$ and $f \vee g \in V$ if $f, g \in V$ (hence also $f \wedge g \in V$) and the collection of open sets $D(f) = f^{-1}(0, \infty)$ form a basis for the topology of X. The next proposition states the existence of partition of unity, without having to mention points [BM3].

Proposition 3.1 *If U_j is an arbitrary covering of X it is possible to find a partition of unity $p_1, \ldots, p_n$ with $p_i \in V$, $0 \leqslant p_i \leqslant 1$ and $\Sigma p_i = 1$ and each open $D(p_i)$ is a formal subset of some U_j.*

Proof. Given any covering U_j we can find positive elements $a_1, \ldots, a_n$ such that the formal open $D(a_i)$ is a formal subset of some U_j and

$$X = D(a_1) \vee \ldots \vee D(a_n) = D(a_1 \vee \ldots \vee a_n)$$

We have then

$$1 \leqslant N(a_1 \vee \ldots \vee a_n) = Na_1 \vee \ldots \vee Na_n$$

for some $N \geqslant 1$. If we define $q_i = 1 \wedge Na_i$ we have thus $\vee q_i = 1$. If we define next $p_i = q_i - (q_i \wedge \vee_{j<i} q_j)$, we have $0 \leqslant p_i \leqslant 1$, each basic open $D(p_i) \subseteq D(a_i)$ is a subset of some U_j and $\Sigma_{j<i} p_j = \vee_{j<i} q_j$. In particular $\Sigma p_i = \vee q_i = 1$. $\square$

Corollary 3.2 *V is dense in $C(X)$.*

We can now recover the density results stated in the references [StoI, StoII].

Theorem 3.3 *If R is an ordered archimedean $\mathbb{Q}$-algebra or Riesz space over $\mathbb{Q}$, the set $\{\hat{a} \mid a \in R\}$ is* dense *in $C(Max(R))$.*

Proof. This is direct from Corollary 3.2 in the case where R is a Riesz space, and in the case of an algebra, this follows also from lemma 1.12. $\square$

Given the results of this paper, it would not be difficult from them to develop Gelfand duality in the real case like in [Joh] but in a constructive way.

4 Integration Algebra

An *integration algebra* [Seg] is a pair (A, E) where A is a $\mathbb{Q}$-algebra and E a linear functional on A such that

$E(a^2) \geqslant 0$

For all elements b there exists c_b such that $E(ba^2) \leqslant c_b E(a^2)$ for all $a \in A$

Segal argues in [Seg] that this is a natural framework in which to develop integration theory, and gives a representation theorem using complex Gelfand duality. We show here that our framework directly gives a representation theorem in the real case. Let (A, E) be an integration algebra. We write $(a, b) = E(ab)$ for $a, b \in R$. We can think now of A as a preHilbert space. In particular, we prove as usual.

Lemma 4.1 *If $a, b \in A$ we have $(a, b)^2 \leqslant (a, a)(b, b)$.*

Each element a of A defines a bounded self-adjoint operator $T_a(b) = ab$ on this space. We let R be the ring of operators generated by the unit operator and the operators $T_a,\ a \in A$. We define a subset P on R by

$$u \in P \ \equiv \ \forall a \in A.0 \leqslant (ua, a)$$

Each operator in R is auto-adjoint and we can prove as usual.

Lemma 4.2 *If $u \in P$ and $(ua, a) \leqslant r(a, a)$ for all $a \in A$ then $(ua, ua) \leqslant r^2(a, a)$ for all $a \in A$.*

We have clearly $u^2 \in P$ for all $u \in R$, and more generally $vu^2 \in P$ if $v \in P$. What is remarkable is the following result.

Proposition 4.3 *If $u \in P$ and $v \in P$ then $uv \in P$*

Proof. (F.Riesz) Let us write $u_1 \leqslant u_2$ iff $u_2 - u_1 \in P$ and, for $u_n \in P$, $u_n \to 0$ iff for all $r > 0$ there exists N such that $u_n \leqslant r$ if $n \geqslant N$.

By axiom 2, we can assume $0 \leqslant v \leqslant 1$.

We define $v_0 = v, v_{n+1} = v_n - v_n^2$. Since

$$v_{n+1} = v_n(1 - v_n)^2 + (1 - v_n)v_n^2 \geqslant 0 \qquad 1 - v_{n+1} = 1 - v_n + v_n^2$$

we have $0 \leqslant v_n \leqslant 1$ for all n. Furthermore $v_n - v_{n+1} = v_n^2$ and hence $v_{n+1} \leqslant v_n$. Also,

$$v_n^2 - v_{n+1}^2 = v_n^2(v_n + v_{n+1})$$

and hence $v_{n+1}^2 \leqslant v_n^2$.

Since $v = v_1^2 + \ldots + v_{n-1}^2 + v_n$ we have $v_n^2 \leqslant v/n$ and so $v_n^2 \to 0$. It follows from lemma 4.2 that $uv_n \to 0$ and since $uv - uv_n = uv_0^2 + \ldots + uv_{n-1}^2 \geqslant 0$ we get $uv \geqslant 0$. □

Theorem 4.4 *If (A, E) is an integration algebra, the set P defined by*

$$u \in P \equiv \forall a \in A. 0 \leqslant (ua, a)$$

is a cone and defines an archimedean preordering on R such that $0 \leqslant u^2$ for all u.

We can thus apply the result of the first part of the paper and consider the formal compact Hausdorff space $Max(R)$, and the elements of R can be thought of as functions on the space $Max(R)$.

For a typical application, if G is a compact abelian group of unit e, and A is the algebra $C(G)$ with the convolution product, and we consider $E(a) = a(e)$, then the open subset $\Sigma = \cup_{a \in A} D(a)$ can be identified with the space of characters over G [Bis]. In this case, each operator T_a is *compact*, and hence each elements of R is normable [Bis]. The next section shows in such a case how to build effectively some points of $Max(R)$, using dependent choice.

5 Positivity on $Max(R)$

We state first a general result on compact completely regular locales. We refer to the [JoO] for a definition of open locales. Intuitively, it means that we have a predicate on open subsets, called positivity predicate, which expresses when an open is inhabited [8].

Theorem 5.1 *If X is a compact completely regular locale, then X is* open *iff for all $f \in C(X)$ there exists sup $f \in \mathbb{R}$ such that sup $f < s$ iff $f(x) < s$ for all $x \in X$*[9].

[8] A basic axiom is that if a positive open is covered by a family, then at least one open in this family should be positive. In particular the empty open is not positive.

[9] To give $f \in C(X)$ is to give two families of open $f^{-1}(-\infty, s)$ and $f^{-1}(r, \infty)$ satisfying some conditions. We write "$f(x) < s$ for all $x \in X$" as a suggestive way to state that $X = f^{-1}(-\infty, s)$.

Proof. In one direction, we define the open $D(f)$ to be positive iff *sup* $f > 0$. It is then direct to check that this defines a positivity predicate.

Conversely, if X is open and $f \in C(X)$ we can find an arbitrary ϵ approximation of the supremum of f by considering a finite covering of X by positive open of the form $f^{-1}(r, s)$, with $s - r < \epsilon$. $\square$

We deduce the following fact, which holds if R is a divisible archimedean ring or a divisible l-group.

Theorem 5.2 *$Max(R)$ is* open *iff for all $f \in R$ there exists sup $f \in \mathbb{R}$ such that sup $f < s$ iff $f \ll s$.*

In the case where $Max(R)$ is open, we can define $||a||$ to be *sup* $a \vee$ *sup* $(-a)$ in $\mathbb{R}$ and the corollary 1.17 gets a sharper version.

Theorem 5.3 *For all a in R, the real $||a||$ is equal to the uniform norm of the map $\hat{a} : Max(R) \to \mathbb{R}, \quad \sigma \longmapsto \sigma(a)$.*

We can also make a connection with the spectral theorem as presented in [Bis]. (It is the only proof which requires the axiom of dependent choice.)

Theorem 5.4 *If R is* separable*, that is contains a dense sequence of elements a_n, and $Max(R)$ is open, for each $f \in R$ such that sup $f > 0$ we can, using dependent choice, find a point $\sigma{:}R \to \mathbb{R}$ of $Max(R)$ such that $\sigma(f) > 0$.*

Proof. Let us write $a \in (p, q)$ for the open $D(q - a) \wedge D(a - p)$ of $Max(R)$. We can find, using dependent choice, $r > 0$ and a sequence $q_n \in \mathbb{Q}$ such that all open sets

$$D(f - r) \wedge a_1 \in (q_1 - 1/2, q_1 + 1/2) \wedge \ldots \wedge a_n \in (q_n - 2^{-n}, q_n + 2^{-n})$$

are positive, that is can be written $D(g)$ with $g \in C(Max(R))$ such that *sup* $g > 0$. If $b \in R$ we can find a subsequence (a_{k_n}) which converges to b. It can then be shown that q_{k_n} converges to a limit l. For this it is enough to notice that if the open

$$a \in (p - r, p + r) \wedge b \in (q - s, q + s)$$

is positive and $|b - a| \leqslant t$ then $|q - p| < r + s + t$. Indeed if $|q - p| \geqslant r + s + t$ then this open is empty and hence cannot be positive. If we take $\sigma(b) = l$ we have defined a function $\sigma{:}R \to \mathbb{R}$ which is a point such that $\sigma(f) > 0$. $\square$

Notice however that it does not mean, even in this case, that the space $Max(R)$ has *enough points* constructively (intuitively the constructive points are recursive and there is not enough recursive points in general). With classical logic and the axiom of choice however, we know that $Max(R)$ being compact regular, has enough points [Joh].

6 f-ring

The structure of f-ring combines the two structures considered by Stone [Bir]. We consider only the case where we have a strong unit 1, in which case the structure can be simply described as an ordered ring which has also a binary sup operation. A typical example is provided by the subsection 1.6.

Lemma 6.1 *In an f-ring we have $ab = 0$ whenever $a \perp b$, and $|a|^2 = a^2$ and $a(b \wedge c) = ab \wedge ac$ if $a \geqslant 0$. If $a, b \geqslant 0$ and $c \perp d$ then $ac \perp bd$.*

Proof. Assume $a \perp b$. We have n such that $a \leqslant n$ and $b \leqslant n$. We have then also $ab \leqslant an$, $ab \leqslant bn$ and since $na \perp nb$ we have $ab = 0$.

If $a \in R$ we have $a = a^+ - a^-$, $|a| = a^+ + a^-$ and $a^+ \perp a^-$. It follows that $a^2 = (a^+)^2 + (a^-)^2 = |a|^2$. □

Corollary 6.2 *We have $(ab)^+ = a^+b^+ + a^-b^-$ and $(ab)^- = a^-b^+ + a^+b^-$.*

Proof. We have $ab = (a^+ - a^-)(b^+ - b^-) = (a^+b^+ + a^-b^-) - (a^-b^+ + a^+b^-)$. Since $a^+ \perp a^-$ and $b^+ \perp b^-$ we have also $a^+b^+ + a^-b^- \perp a^-b^+ + a^+b^-$, hence the result. □

Lemma 6.3 *We have $(a - r)^+ \wedge b^+ \leqslant 1/r(ab)^+$ if $r > 0$.*

Proof. Using the corollary 6.2 we reduce this to $(a - r)^+ \wedge b^+ \leqslant 1/ra^+b^+$. Writing $u = (a - r)^+ \wedge b^+$ this in turn follows from $ru \leqslant a^+u$ or $0 \leqslant u(a^+ - r)$. This holds since $u(a^+ - r)^- = 0$ because $u \leqslant (a^+ - r)^+$ and lemma 6.1. □

Theorem 6.4 *Let R be an f-ring with a strong unit. In the lattice $Tot(R)$ the schema*

$$D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$$

implies

$$D(a \vee b) = D(a) \vee D(b)$$

In the other direction the schema

$$D(a \vee b) = D(a) \vee D(b)$$

together *with the continuity axiom $D(a) = \bigvee_{r>0} D(a - r)$ implies*

$$D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$$

Proof. Assume

$$D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$$

Notice that this implies $D(a) \wedge D(b) = 0$ if $ab = 0$. By the proposition 2.6 it is enough to show $D(a^+) = D(a)$. We have $D(a^2) = D(a) \vee D(-a)$. Hence in particular $D(x^2) = D(x)$ if $0 \leqslant x$. By the lemma 2.2

$$D(a^2) = D(|a|^2) = D(|a|) \leqslant D(a^+) \vee D(a^-)$$

since $D(a^+) \leqslant D(|a|)$, $D(a^-) \leqslant D(|a|)$ it follows that we have

$$D(|a|) = D(a^+) \vee D(a^-) = D(a) \vee D(-a)$$

We have $a^+a^- = 0$ by lemma 2.2 and hence $D(a^+) \wedge D(a^-) = 0$. Since $D(a) \leqslant D(a^+)$ and $D(-a) \leqslant D(a^-)$ it follows that we have $D(a) = D(a^+)$, hence the result.

Conversely, assume the continuity axiom and $D(a \vee b) = D(a) \vee D(b)$. We use the lemma 1.4 and prove $D(a) \wedge D(b) \leqslant D(ab)$ and $D(ab) \leqslant D(a) \vee D(-b)$.

Using the theorem 2.7 we reduce for each $r > 0$

$$D(a - r) \wedge D(b - r) \leqslant D(ab)$$

to the inequality $(a - r)^+ \wedge b^+ \leqslant 1/r(ab)^+$ which is lemma 6.3. By continuity this implies $D(a) \wedge D(b) \leqslant D(ab)$. We show next $D(ab) \leqslant D(a) \vee D(-b)$ using the fact that we have a strong unit: there exists n such that $|a| \leqslant n$ and $|b| \leqslant n$. It is then direct that we have, using corollary 6.2

$$(ab)^+ = a^+b^+ + a^-b^- \leqslant n(a^+ + b^-)$$

which by the theorem 2.7 implies $D(ab) \leqslant D(a) \vee D(-b)$ □

Conclusion

In physical terms, both algebraic structures, ordered ring and l-group, cover the case of a system of real, simultaneously observable physical quantities as envisaged in the quantum theory. It would be interesting to compare in the present constructive framework the generalisation of these two algebraic structures in the case where the quantities represented by the elements of the structure may not be always simultaneously observable. In the ring case, one takes away the commutativity axiom, and considers that two quantities are simulatenously observable iff the corresponding operators commute. In the l-group case, one has to take away the lattice axioms, and considers that two quantities are simulatenously observable iff the corresponding operators have a least upper bound.

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