Ω-rule

Content of the course

Part I: negative translation

Part II: inductive definitions, system ID_1,$\mathrm{ID}_2, \ldots$ and ω-rule

Part III: Buchholz' Ω-rule

Part IV: impredicative definitions, Π^1_1 comprehension axiom

Two main results in proof theory:

Reduction of Π^1_1-comprehension to inductive definitions (Takeuti)

Reduction of the *classical* theory of inductive definitions to the *intuitionistic* theory of inductive definitions

Hilbert's program

Debate on foundations of mathematics: Hilbert, Brouwer

A typical example: the proof by Hilbert's of the finite basis theorem

Hilbert's proof starts by showing that if we have a sequence of natural numbers $n_0, n_1, n_2, \ldots$ it has a minimum $n_k \leq n_p$ for all p

It is impossible to give explicitely p as a computable function of the sequence $n_0, n_1, n_2, \ldots$

According to Brouwer, this kind of proof gives *no* mathematical insight. Before, Kronecker had argued that this kind of argument should be avoided with profit in algebra.

Hilbert's viewpoint: this kind of argument *simplifies* mathematics and allows us to get conceptual insight

Hilbert's program

We describe Hilbert's strategy (see R. Zach's papers)

First, quantifier-free theory, atomic propositions are equations and the basic axioms are

$$x + 1 \neq 0 \qquad \delta(x+1) = x$$

$$x = y \rightarrow A(x) \rightarrow A(y)$$

we can add new definitions of functions, such as

$$x + 0 = x \qquad x + (y+1) = (x+y) + 1$$

$$x \cdot 0 = 0 \qquad x \cdot (y+1) = (x \cdot y) + x$$

$$g(0,y) = y{+}1 \qquad g(x{+}1,0) = g(x,1) \qquad g(x{+}1,y{+}1) = g(x,g(x{+}1,y))$$

Hilbert's program

This system is *consistent*

The proof of consistency is simple: if we replace variables by numerals we can compute the truth-value of each atomic statements and so $1 \neq 0$ is not provable.

Hilbert's program

Next step: to extend the system with a non computable operator $\tau_x(t)$, specified by the axiom

$$t(x = \tau_x(t)) = 0 \quad \to \quad t = 0$$

In general, it is not possible to compute $\tau_x(t)$ so the simple proof of consistency does not work anymore

Instead one tries to show that, in any given proof, one can eliminate the use of this non computable operator

The method is quite interesting: try to guess the value of $\tau_x(t)$ and use the proof to make other guesses and "learn" about this value

We shall find a similar idea in Gentzen's ω-rule in the second lecture

Hilbert's program

Brouwer thought that this would be possible but even if it worked, would not be useful: "An incorrect theory which is not stopped by a contradiction is none the less incorrect, just as a criminal policy unchecked by a reprimanding court is none the less criminal."!!

Hilbert's program

The general theme of these lectures will be

explanation of non-constructive principles (excluded-middle, non computable functions) in a constructive setting

This was what Hilbert, and then Gentzen were trying to do

"Thus propositions of actualist mathematics seem to have a certain utility, but no *sense*. The major part of my consistency proof, however, consists precisely in *ascribing a finitist sense* to actualist propositions." (Gentzen)

Negative translation

Kolmogorov 1925 provided a general solution to Hilbert's problem by giving an intuitionistic explanation of the law of excluded-middle

Kolmogorov saw clearly that this works for a large fragment of mathematics (but gave only a complete treatment for propositional and first-order logic)

Later (1933) Gödel published a complete treatment as a translation from Peano arithmetic to Heyting arithmetic

Gentzen had, independently, a similar treatment

Negative Translation

The main idea is captured by the following dialogue (due to E. Nelson)

C: I just proved $\exists x.A$.

I: Congratulations! What is it?

C: I don't know. I assumed $\forall x.\neg A$ and derived a contradiction.

I: I see. You proved $\neg\forall x.\neg A$.

C: Yes, that's what I said.

Negative Translation

C: I just proved $A \vee B$.

I: Good. Which did you prove?

C: What?

I: You said you proved $A \vee B$; which did you prove?

C: Neither. I assumed $\neg A \wedge \neg B$ and derived a contradiction.

I: Oh, you proved $\neg[\neg A \wedge \neg B]$.

C: That's right. It's another way of putting the same thing.

"But he does not agree with her last statement; they have a different semantics and a different notion of proof."

Negative translation

The intuitionist mathematician can make sense of what the classical mathematician is doing, provided he transforms a little the statements

$\exists x.A$ is interpreted as $\neg\forall x.\neg A^*$

$A \vee B$ is interpreted as $\neg\ (\neg A^* \wedge \neg B^*)$

The interpretation commutes with other connectives and quantifications

$\forall x.A$ is interpreted as $\forall x.A^*$

$A \wedge B$ is interpreted as $A^* \wedge B^*$

$A \rightarrow B$ is interpreted as $A^* \rightarrow B^*$

$\neg A$ is interpreted as $\neg A^*$

Negative translation

There is some choice: Gödel interpreted $A \to B$ as $\neg(A^* \wedge \neg B^*)$

The interpretation of Kolmogorov was not minimal, but more systematic

$A \wedge B$ becomes $\neg\neg(A^* \wedge B^*)$

$A \to B$ becomes $\neg\neg(A^* \to B^*)$

$A \vee B$ becomes $\neg\neg(A^* \vee B^*)$

$\neg A$ becomes $\neg\neg(\neg A^*)$

In general

$F(A_1, \ldots, A_n)$ becomes $\neg\neg F(A_1^*, \ldots, A_n^*)$

Negative translation

Gentzen's version of negative translation

$A^* \equiv A$ if A atomic

$(\forall x.A)^* \equiv \forall x.A^*$

$(A \wedge B)^* \equiv A^* \wedge B^*$

$(A \to B)^* \equiv A^* \to B^*$

$(\neg A)^* \equiv \neg A^*$

$(\exists x.A)^* \equiv \neg \forall x.\neg A^*$

$(A \vee B)^* \equiv \neg\ (\neg A^* \wedge \neg B^*)$

Negative translation

For all these interpretations, $(A \vee \neg A)^*$ is provable intuionistically

Furthermore if A^* and $(A \to B)^*$ are provable, then so is B^*. Also, if $(\forall x.A)^*$ is provable then so is $A(t)^*$

Thus to interpret classical reasoning, starting from axioms, we need only to check that the translations of these axioms are intuitionistically provable

Peano Arithmetic

Introduced in Gödel's paper (after Herbrand)

This is formulated in first-order logic. The core axioms are

$$x+1 \neq 0 \qquad x+1 = y+1 \rightarrow x = y$$

This is a first-order theory with no finite model, hence the consistency cannot be established finistically by exhibing a model

von Neumann (1927) showed the consistency of this theory, using the operator τ

This was much simplified by Herbrand, by using quantifier eliminations and added also

$$x = 0 \vee \exists y. x = y+1$$

Peano Arithmetic

In Hilbert's system we add a function δ with the axiom

$$\delta(x+1) = x$$

Then

$$x + 1 = y + 1 \rightarrow x = y$$

is provable by substitutivity, as well as

$$x = 0 \vee \exists y. x = y + 1$$

Peano Arithmetic

It was noticed by von Neumann that one can add as well the axiom schema

$$A(0) \wedge (\forall n. A(n) \rightarrow A(n+1)) \rightarrow \forall x. A$$

This was proved by the technique of relativisation: if one uses the axiom schema on $A_1, \ldots, A_k$ one can relativise the statements using $A(x) = \wedge A_i(x)$ and

$$N(x) \equiv \quad A(0) \wedge (\forall n. A(n) \rightarrow A(n+1)) \rightarrow A(x)$$

Peano Arithmetic

The theory becomes undecidable (and one cannot use quantifier elimination to show its consistency) if one adds

$$x + 0 = x \qquad x + (y + 1) = (x + y) + 1$$

$$x \cdot 0 = 0 \qquad x \cdot (y + 1) = (x \cdot y) + x$$

Gödel, following Herbrand and Hilbert, considered an “open” theory where one can add new functions with new schemas, like

$$f(0) = 1 \qquad f(x + 1) = f(x) + f(x)$$

$$g(0, y) = y{+}1 \qquad g(x{+}1, 0) = g(x, 1) \qquad g(x{+}1, y{+}1) = g(x, g(x{+}1, y))$$

Peano Arithmetic

Each closed atomic formulae $t = u$ is *decidable* like for Hilbert's systems

The non effectivity comes from universal and existential quantification combined with classical logic, and not from the introduction of non computable functionals

Gödel's 1933 paper shows: it is possible (and relatively easy) to show the consistency of PA in an intuitionistic way!!

Negative translation of PA

We denote by HA the theory with the same axioms as PA but which uses intuitionistic logic instead of classical logic, PA = HA + EM

We define $A^* = A$ if A is atomic

We prove $x = y \vee \neg(x = y)$ in HA

All axioms are translated to themselves except the induction axioms

But

$$A^*(0) \wedge (\forall n.A^*(n) \rightarrow A^*(n+1)) \rightarrow \forall x.A^*$$

is an *instance* of the induction schema, so any proof of PA can be translated in a proof in HA

Hence PA is consistent!

Negative translation

Gödel in his incompleteness paper says explicitely that his result does not contradict Hilbert's program, because there may be finitary methods that cannot be represented in the system of Principia Mathematica

Negative translation "explains" PA to an intuitionistic mathematician so it is an intuitionistic proof of consistency of PA

Negative translation

In 1933, Gödel changed his mind: HA is not "finitary", so we have an *intuitionistic* proof of consistency, which is not *finitary*

E. Nelson's interpretation of the negative translation is that it shows that intuitionistic mathematics is *not* a restriction of classical mathematics, but an *extension*. It completes classical mathematics by adding one new connective $A \vee B$ and one new quantification $\exists x.A$ (effective existence) that do not exist in classical mathematics.

Negative translation of PA

General method, observed by Kolmogorov: in order to justify a classical principle A in term of its intuitionistic version, it is enough to check that the negative translation A^* is an intuitionistic consequence of the principle A

Extension

The negative translation works for

- second-order arithmetic: we have also predicate variables and quantification over predicates
- second-order logic
- simple type theory
- set theory

According to Gödel's 1933 paper, impredicative definitions are *not* justified intuitionistically

One of the main problem in proof theory has been to explain intuitionistically some restricted impredicative definitions (we shall provide such an explanation in the last lecture, using the Ω-rule)

Refinements

The negative translation is most interesting viewed through Curry-Howard correspondance between proof and programs

Friedman (and independently Dragalin) use a variation to show the following: if PA$\vdash \forall x \exists y.A$, where A is quantifier-free, then HA$\vdash \forall x \exists y.A$. Thus a classically provable Π^0_2 statement is intuitionistically provable.

This is *not* valid for statements of the form $\forall x \exists y \forall z.A(x, y, z)$ It may may provable in PA but that there is *no* computable function f such that for all n the statement $\forall z.A(n, f(n), z)$ is valid

Natural question: what does the statement $\forall x \exists y \forall z.A(x, y, z)$ mean classically?? This is what Gentzen has found and will be analysed in the second lecture.

Case where the negative translation does not work

The system HA^ω is like HA but we can quantify over functions, functionals

The axiom of choice AC if

$$\forall x \exists y. A(x, y) \to \exists f \forall x. A(x, f(x))$$

For proving AC^* from AC, one needs $(\forall x. \neg\neg A) \to \neg\neg\forall x. A$ which is not valid intuitionistically

Recently, U. Berger showed that one can replace the axiom of choice by a classically equivalent principle: the open induction, which has the property of implying its negative translation.

The system HA^ω + AC + EM is strictly stronger proof theoretically than the system HA^ω + AC (which has the same strength as HA)

Case where the negative translation does not work

Arithmetic with Σ^0_1 induction

Theory ID^c_1 (presented in the second/third lecture)

There the situation is more subtle: the negative translation does not work, but it is possible to give a translation of the classical version in the corresponding intuitionistic system

References

Gödel Collected Work, IV and V, especially Herbrand, von Neumann, Bernays

One main question seems to be if one can bound a priori the proof theoretic strength of intuitionistic mathematics

Metamathematical investigation of intuitionistic arithmetic and analysis. Edited by A. S. Troelstra. Lecture Notes in Mathematics, Vol. 344. Springer-Verlag, Berlin-New York, 1973.

See also the papers of E. Nelson in his home page at http://www.math.princeton.edu/~nelson/papers.html

and of R. Zach in his home page at http://www.ucalgary.ca/~rzach/papers/conprf.html

Exercices

Show

$$\neg\neg A \wedge \neg\neg B \quad \rightarrow \quad \neg\neg(A \wedge B)$$

and notice that there are essentially two different proofs of this

Explain why

$$\forall x.\neg\neg A(x) \quad \rightarrow \quad \neg\neg\forall x.A(x)$$

is not valid intuitionistically