Impredicative Definitions

One main goal of proof theory, since the debate between Poincaré and Russell, has been to analyse impredicative definitions

Typically: a real number x is defined as a set of rationals q such that $q < x$. Given a formula $\phi(X)$, for instance

$$(\exists q.X(q) \land q > 0) \land \forall q_1, q_2.q_1 < q_2 \rightarrow X(q_2) \rightarrow X(q_1)$$

the g.l.b. of the collection of all X such that $\phi(X)$ is given by the predicate

$$\forall X.\phi(X) \rightarrow X(q)$$

This predicate is defined by quantification over all possible predicates

Impredicative Definitions

Such a definition looks circular (Poincaré)

In this case, the predicate can be rewritten as $q \leq 0$, so the circularity is only apparent

Impredicative Definitions

Takeuti formulated, in the 50s, a sequent calculus for second-order arithmetic, and conjectured cut-elimination

He could prove cut-elimination for a restricted version to Π^1_1-comprehension

Kreisel, by analysing the proof in a review, noticed that the argument can be represented in an intuitionistic system of inductive definitions

Impredicative Definitions

Buchholz found a variation of the Ω-rule that allows a more direct interpretation of Π^1_1-comprehension in term of inductive definitions

A particularly simple version of this reduction is obtained by showing the normalisation of a restricted fragment of system F with only quantification over finite objects

Impredicative Definitions

One main intuition can be found in Lorenzen (1958): it is possible to explain the classical truth of a statement

$$\forall X.\phi(X)$$

where ϕ does not have any quantification on predicates, by saying that

$$\phi(X)$$

is classically valid, where X is a variable

We know how to express this using inductive definitions

For instance, it can be seen in this way that

$$\forall X.X(5) \rightarrow X(5)$$

is valid, without having to consider all subsets of $\mathbb{N}$

Impredicative Definitions

To take another example, with

$$\phi(X) \equiv \quad (\exists q. X(q) \wedge q > 0) \wedge \forall q_1, q_2. q_1 < q_2 \rightarrow X(q_2) \rightarrow X(q_1)$$

it is possible to show directly that

$$\vdash \phi(X) \rightarrow X(q)$$

is provable in ω-logic, with X variable predicate, iff $q \leq 0$.

Furthermore this reasoning will only involve inductive definitions, and not the explicit consideration of all subsets of $\mathbb{Q}$

Impredicative Definitions

If

$$\vdash X(q_0), \forall q > 0. \neg X(q), \exists p < q. X(q) \wedge \neg X(p)$$

is provable then, by inversion

$$\vdash X(q_0), \neg X(q_1), \exists p < q. X(q) \wedge \neg X(p)$$

is provable for each $q_1 > 0$

It is direct to see that if $0 < q_1 < q_0$ then

$$\vdash X(q_0), \neg X(q_1), \exists p < q. X(q) \wedge \neg X(p)$$

is not provable

A subsystem of system F

System F was introduced by J.Y. Girard (1970) for giving a Dialectica interpretation of second-order arithmetic

One can show the normalisation of system F, but only by using in the meta-language impredicative definitions

$$T \quad ::= \quad \alpha \mid T \rightarrow T \mid (\Pi\alpha)T$$

Each close type can be interpreted in a natural, but impredicative way, as a set of *untyped* λ-terms

It is then direct to show that all terms in such a set are normalisable

A subsystem of system F

This can be interpreted as a normalisation theorem for the following typing rules

$$\frac{}{\Gamma \vdash x : T} \quad x : T \in \Gamma$$

$$\frac{\Gamma, x : T \vdash t : U}{\Gamma \vdash \lambda x \ t : T \rightarrow U} \qquad \frac{\Gamma \vdash u : V \rightarrow T \quad \Gamma \vdash v : V}{\Gamma \vdash u \ v : T}$$

$$\frac{\Gamma \vdash t : (\Pi\alpha)T}{\Gamma \vdash t : T[U]} \qquad \frac{\Gamma \vdash t : T}{\Gamma \vdash t : (\Pi\alpha)T}$$

where Γ is a finite set of type declaration $x : T$, and in the last rule, α does not appear free in any type of Γ.

A subsystem of system F

We consider the following types

$$T \quad ::= \quad \alpha \mid T \to T \mid (\Pi\alpha)T$$

where in the quantification, T has to be built using only α and $\to$.

Then the normalisation theorem can be shown without impredicative definitions

A subsystem of system F

General strategy: we define a Kripke model using only finite objects

We interpret the usual proof of normalisation, interpreting each type as an H-valued predicate

We build H in such a way that, relative to this model, each impredicatively defined predicate required for this proof is *equivalent* to a predicate defined using only quantifications on finite objects

References

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