Proof Theory in Type Theory

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Introduction

The negative translation provides a general way to make constructive sense of some non effective reasoning. However this method has some limitations. It does not work in presence of the axiom of description/choice. In this note, we analyse the interaction of classical logic with generalised inductive definition. In the metatheory, we allow generalised inductive definitions, but with an intuitionistic logic. Actually, we can take as our metatheory a system like Martin-Löf type theory.

The starting motivation of this work was a comparison between the work of Tait/Schütte and the work of Lorenzen/Novikov on cut-elimination. From a constructive perspective, the analysis of sequent calculus is most elegantly expressed in Lorenzen/Novikov's way, as the *admissibility* of the cut rule [2, 3]. Tait/Schütte's approach introduces two extra parameters $\vdash^{\alpha}_{\rho} \Gamma$ which measures the depth of the proof tree and the maximal degree of the cut in the proof (see for instance [1]). While this may be extremely interesting for an evaluation of the complexity of the cut-elimination processes, if the goal is only to give a constructive explanation of some classical reasoning, these extra parameters should be irrelevant.

Looking at the presentation of local predicativity in [1, 4], one may wonder if this is indeed the case.

We present a Lorenzen/Novikov version of two methods of analysis of inductive definitions with classical logic. One is local predicativity due to W. Pohlers, and the other uses the so called Ω-rule and is due to W. Bucccholz . Like in the book [4], we limit ourselves to the simplest case of classical system of generalised inductive definitions. We are aware that we loose some of the detailed informations that we can get by this method [1]. The hope is however that this note will make these methods more accessible, especially to people familiar with type theory and inductive definitions.

1 A Theory of Inductive Definitions

Language: a first-order language over natural numbers and lists of natural numbers. We have a decidable property $\mathsf{G}(\sigma)$ over lists and we use $\mathsf{B}(\sigma)$ to express that the finite list σ is barred by B.

Let ID be the *classical* theory of B, which expresses that B is the least predicate containing G such that $\mathsf{B}(\sigma)$ holds whenever $\mathsf{B}(\sigma n)$ holds for all n. For the details, see for instance [4].

The goal of this note is to present a proof of the *existence property* of the system ID, that is that any Σ^1_0 formula provable in ID is valid intuitionistically.

This result is not obvious since ID claims in particular $\mathsf{B}(\sigma) \vee \neg\mathsf{B}(\sigma)$ while the property B is not, a priori, decidable, even if the property G is. What is interesting here is that the usual method of *negative translation* does not seem to work: in order to interpret the introduction rule of B, one would need the implication

$$\forall n \neg\neg \mathsf{B}(\sigma n) \;\rightarrow\; \neg\neg\forall n \mathsf{B}(\sigma n)$$

which is not intuitionistically valid in general, since B is not decidable.

In order to achieve this goal, we shall embed ID in an infinitary propositional calculus. The formulae of this calculus are defined inductively. The atomic formulae are $0,\ 1$ and $\mathsf{B}(\sigma), \mathsf{B}'(\sigma)$, where σ is a list of natural numbers. The composed formulae are $\bigvee A_i$ and $\bigwedge A_i,\ i \in I$ where I is either a finite set, or the

[1]However, in the case of the Ω-rule, it seems clear intuitively how to recover the missing informations.

set of natural numbers N or the set L of list of natural numbers. The negation is defined recursively by the de Morgan rules: we have $\neg \bigwedge A_i = \bigvee \neg A_i$ and $\neg \bigvee A_i = \bigwedge \neg A_i$, and $\neg \mathsf{B}(\sigma) = \mathsf{B}'(\sigma)$, $\neg \mathsf{B}'(\sigma) = \mathsf{B}(\sigma)$.

We will express B as a cumulative union in term of G. We start by $\mathsf{G}^0(\sigma) = \mathsf{G}(\sigma)$ and then $\mathsf{G}^{\alpha+1}(\sigma) = \mathsf{G}^{\alpha}(\sigma) \vee \forall n \mathsf{G}^{\alpha}(\sigma n)$, and $\mathsf{G}^{\alpha}(\sigma) = \bigvee_k \mathsf{G}^{u(k)}(\sigma)$ if $\alpha = \mathbf{sup}(u)$.

Notice that the equality on formulae is not decidable. A formula is *positive* iff it has no occurences of an atomic formula of the form $\mathsf{B}'(\sigma)$. We let Pos denote the set of all positive formulae.

It is direct how to translate a formula A of ID in a formula A^* of the present infinitary propositional calculus. For instance, the formula $\forall \sigma [\exists n \mathsf{B}(\sigma n) \vee \neg \exists n \mathsf{B}(\sigma n)]$ is translated in the formula $\bigwedge_\sigma [\bigvee_n \mathsf{B}(\sigma n) \vee \bigwedge_n \mathsf{B}'(\sigma n)]$.

2 A Partial Cut Elimination

The *sequents* are finite sets of the infinitary propositional calculus. We write Γ, A for $\Gamma \cup \{A\}$ and Γ, Δ for $\Gamma \cup \Delta$. A seqyent is *positive* iff it contains only positive formulae. We shall build a sequent calculus (S_1) such that, if A is provable in ID then $\vdash A^*$ in (S_1). Furthermore, we show that if A_n is a sequence of decidable formulae, and $\vdash \bigvee A_i$ in (S_1) then there exists i_0 such that A_{i_0} is true.[2] We establish in this way existence property of ID.

Ideally, we would like to show that if Γ is provable, then it is provable without cuts. This can be reformulated as the fact that the cut rule is *admissible*. What we find interesting here is that this is shown *only* if there is no negative occurence of B is Γ.

2.1 A First Sequent Calculus

We introduce first a sequent calculus (S_0) with the rules

1. $\vdash \Gamma$ whenever 1 is in Γ,
2. $\vdash \Gamma, A_{i_0} \;\; \rightarrow \;\; \vdash \Gamma$ whenever $\bigvee A_i$ is in Γ,
3. $[\forall i \vdash \Gamma, A_i] \;\; \rightarrow \;\; \vdash \Gamma, \bigwedge A_i$,
4. $\vdash \Gamma, \mathsf{G}^{\alpha}(\sigma) \;\; \rightarrow \;\; \vdash \Gamma$ whenever $\mathsf{B}(\sigma)$ is in Γ,
5. $[\forall \alpha \vdash \Gamma, \neg \mathsf{G}^{\alpha}(\sigma)] \;\; \rightarrow \;\; \vdash \Gamma, \mathsf{B}'(\sigma)$.

Alternatively, one may think of a proof in (S_0) as a tree built from the following rules.

$$\overline{\Gamma, 1}$$

$$\frac{\vdash \Gamma, \bigvee A_i, A_{i_0}}{\vdash \Gamma, \bigvee A_i}$$

$$\frac{\ldots \vdash \Gamma, A_i \ldots}{\vdash \Gamma, \bigwedge A_i}$$

$$\frac{\vdash \Gamma, \mathsf{B}(\sigma), \mathsf{G}^{\alpha}(\sigma)}{\vdash \Gamma, \mathsf{B}(\sigma)}$$

$$\frac{\ldots \vdash \Gamma, \neg \mathsf{G}^{\alpha}(\sigma) \ldots}{\vdash \Gamma, \mathsf{B}'(\sigma)}.$$

Notice that the branching of this tree may be (intuitionistically) uncountable if the last rule is used.

Remark 1: If $\Gamma \subseteq \Delta$ and $\vdash \Gamma$ then $\vdash \Delta$.

Remark 2: For any formula A we have $\vdash A, \neg A$.

Lemma 1: The cut rule is admissible for (S_0).

[2] Notice that this means in particular that we can find such a i_0 explicitely, since our metalanguage is constructive.

Proof: This means that we have

$$\vdash \Gamma, A, \quad \vdash \Delta, \neg A \ \rightarrow \ \vdash \Gamma, \Delta;$$

the proof is similar to the usual proof of admissibility of cut, by doing an induction first on the cut formula A, and then on the proofs of $\vdash \Gamma, A$ and $\vdash \Delta, \neg A$ (see [2, 3]). $\square$

The following lemma is a kind of partial inversion for the introduction rule of $\mathsf{B}(\sigma)$.

Lemma 2: If $\vdash \Gamma, \mathsf{B}(\sigma)$ in (S_0) and Γ is positive then $\vdash \Gamma, \mathsf{G}^{\alpha}(\sigma)$ in (S_0) for one α.

Proof: In the case of the closure rule, we have by induction hypothesis a sequence α_n such that $\vdash \Gamma, \mathsf{G}^{\alpha_n}(\sigma n)$ in (S_0) for all n. It follows that $\vdash \Gamma, \mathsf{G}^{\alpha}(\sigma n)$ for all n, where $\alpha = \mathbf{sup}(\alpha_n)$. Hence we have $\vdash \Gamma, \bigwedge_n \mathsf{G}^{\alpha}(\sigma n)$ and hence $\vdash \Gamma, \mathsf{G}^{\alpha+1}(\sigma)$ in (S_0).

Since Γ is positive, the other difficult case is that $\vdash \Gamma, \mathsf{B}(\sigma)$ follows from a sequence $\vdash \Gamma_n, \mathsf{B}(\sigma)$ where each Γ_n is positive. By induction hypothesis, we have $\vdash \Gamma_n, \mathsf{G}^{\alpha_n}(\sigma)$ in (S_0) for some α_n and hence $\vdash \Gamma_n, \mathsf{G}^{\alpha}(\sigma)$ for all n, where $\alpha = \mathbf{sup}(\alpha_n)$. This implies $\vdash \Gamma, \mathsf{G}^{\alpha}(\sigma)$ in (S_0). $\square$

The formal system PA can be embedded in infinitary propositional calculus; the key point there is that the induction axiom becomes provable. Surprisingly, the same does not hold for the system ID, which cannot be embedded in (S_0). Furthermore, the problem is in the following introduction rule of ID

$$[\forall n \mathsf{B}(\sigma n)] \ \rightarrow \ \mathsf{B}(\sigma).$$

The translation in (S_0) would be the sequent $\vdash \mathsf{B}(\sigma), \bigvee_n \mathsf{B}'(\sigma n)$ which does not seem to be provable in (S_0).

2.2 The closure rule

We need thus to extend the system (S_0) in such a way that this sequent becomes provable. One way to do it is to add the following rule, called the *closure rule*

$$[\forall n \vdash \Gamma, \mathsf{B}(\sigma n)] \ \rightarrow \ \vdash \Gamma, \mathsf{B}(\sigma).$$

From this, it follows that (S_1) is strong enough to interpret the *classical* theory of the predicate B. In particular, the sequent $\vdash \mathsf{B}(\sigma), \bigvee_n \mathsf{B}'(\sigma n)$ is provable by the closure rule, since we can show for all n_0 $\vdash \mathsf{B}(\sigma n_0), \bigvee_n \mathsf{B}'(\sigma n)$ in (S_0) (and hence in (S_1)).

However, the cut rule does not seem to be admissible for this extended system and we have to add also the cut rule restricted on formulae of the form $\mathsf{B}(\sigma)$

$$\vdash \Gamma, \mathsf{B}(\sigma), \quad \vdash \Delta, \mathsf{B}'(\sigma) \ \rightarrow \ \vdash \Gamma, \Delta.$$

Lemma 3: The cut rule on *any* formula is admissible for (S_1).

Proof: As the usual proof of admissibility of cut. $\square$

However, it seems hard to remove the cut rule on formula $\mathsf{B}(\sigma)$ from the system (S_1). At this point, we cannot conclude the existence property of ID. We are going to show that, if Γ is positive then these cut rules can be eliminated from a proof of $\vdash \Gamma$ in (S_1).[3]

From the admissibility of the general cut rule in (S_1), it is not hard to conclude

Corollary: If A is derivable in ID then $\vdash A^*$ in (S_1), where A^* is the translation of A in infinitary propositional calculus.

[3]It has to be noted that Tait has a similar restriction in his analysis of Σ^1_1-AC [5].

2.3 Partial Cut Elimination

We can now state our main lemma:

Lemma 4: If $\Gamma \subseteq \mathsf{Pos}$ and $\vdash_{S1} \Gamma, \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k)$ then $\vdash_{S_0} \Gamma, \neg\mathsf{G}^{\alpha_1}(\sigma_1), \ldots, \neg\mathsf{G}^{\alpha_k}(\sigma_k)$ for all $\alpha_1, \ldots, \alpha_k$.

Proof: We look at the case of a cut rule, when $\vdash \Gamma, \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k)$ follows from[4]

$$\vdash \Gamma, \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k), \mathsf{B}(\sigma)$$

and

$$\vdash \Gamma, \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k), \mathsf{B}'(\sigma).$$

By induction hypothesis, for all $\alpha_1, \ldots, \alpha_k, \alpha$ we have

$$\vdash \Gamma, \mathsf{B}(\sigma), \neg\mathsf{G}^{\alpha_1}(\sigma_1), \ldots, \neg\mathsf{G}^{\alpha_k}(\sigma_k)$$

and

$$\vdash \Gamma, \neg\mathsf{G}^{\alpha}(\sigma), \neg\mathsf{G}^{\alpha_1}(\sigma_1), \ldots, \neg\mathsf{G}^{\alpha_k}(\sigma_k).$$

By lemma 2, we have

$$\vdash \Gamma, \mathsf{G}^{\alpha_0}(\sigma), \neg\mathsf{G}^{\alpha_1}(\sigma_1), \ldots, \neg\mathsf{G}^{\alpha_k}(\sigma_k)$$

for one α_0 and hence, by lemma 1,

$$\vdash \Gamma, \neg\mathsf{G}^{\alpha_1}(\sigma_1), \ldots, \neg\mathsf{G}^{\alpha_k}(\sigma_k)$$

in (S_0). $\square$

Theorem: If $\Gamma \subseteq \mathsf{Pos}$ and $\vdash \Gamma$ in (S_1) then $\vdash \Gamma$ in (S_0).

Proof: This is a particular case of lemma 4.

Corollary: If each A_i is 0 or 1, and $\vdash \bigvee A_i$ in (S1) then there exists i_0 such that A_{i_0} is 1.

3 The Ω-rule

It turns out that another approach, the Ω-rule, due to Buccholz [1], is actually more appropriate to a representation in Type Theory. Interestingly, the two methods do not look so far apart when formulated in a type theoretical framework.

In this formulation we don't need to consider ordinals and formulae G^α. The system S_0 has now for rules

$$\frac{}{\Gamma, 1}$$

$$\frac{}{\Gamma, \mathsf{B}(\sigma)} \ (\text{if } \mathsf{G}(\sigma))$$

$$\frac{\vdash \Gamma, \bigvee A_i, A_{i_0}}{\vdash \Gamma, \bigvee A_i}$$

$$\frac{\ldots \vdash \Gamma, A_i \ldots}{\vdash \Gamma, \bigwedge A_i}$$

together with the closure rule

$$[\forall n \vdash \Gamma, \mathsf{B}(\sigma n)] \ \rightarrow \ \vdash \Gamma, \mathsf{B}(\sigma).$$

The system (S_1) has for rules the restricted cut rule

$$\vdash \Gamma, \mathsf{B}(\sigma), \quad \vdash \Delta, \mathsf{B}'(\sigma) \ \rightarrow \ \vdash \Gamma, \Delta$$

[4] In general, it will follow from $\vdash \Delta, \mathsf{B}(\sigma)$ and $\vdash \Theta, \mathsf{B}'(\sigma)$ with Δ, Θ is $\Gamma, \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k)$. But this general case can be treated in the same way as the case we present here.

and the remarkable Ω-rule:

$$\forall \Delta \subseteq \mathsf{Pos}[\vdash_{S_0} \Delta, \mathsf{B}(\sigma) \ \rightarrow \ \vdash_{S1} \Delta, \Gamma] \ \rightarrow \ \vdash_{S_1} \Gamma, \mathsf{B}'(\sigma).$$

Intuitively, we use arbitrary proof of $\vdash \mathsf{B}(\sigma)$ instead of using ordinals.

For instance, we have clearly

$$\forall \Delta \subseteq \mathsf{Pos}[\vdash_{S_0} \Delta, \mathsf{B}(\sigma) \ \rightarrow \ \vdash_{S1} \Delta, \mathsf{B}(\sigma)]$$

because S_0 is a subsystem of S_1 and hence, by the Ω-rule

$$\vdash_{S_1} \mathsf{B}(\sigma), \mathsf{B}'(\sigma).$$

It follos by the usual argument that we have $\vdash_{S_1} A, \neg A$ for any formula A.

It is then direct to prove

Proposition 1: The cut rule on any formula is admissible in (S_1).

Proof: This is direct by induction first on the cut formula, and then on the two derivations.

It follows from this that we can embed (ID) in the new system (S_1).

Proposition 2: If $\Gamma \subseteq \mathsf{Pos}$ then Γ is provable in (S_1) iff it is provable in (S_0).

Lemma: If $\Gamma \subseteq \mathsf{Pos}$ and $\vdash_{S_1} \Gamma, \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k)$ then for any $\Delta_1, \ldots, \Delta_k \subseteq \mathsf{Pos}$ if we have $\vdash_{S_0} \Delta_1, \mathsf{B}(\sigma_1), \ldots, \vdash_{S_0} \Delta_k, \mathsf{B}(\sigma_k)$ we also have $\vdash_{S_0} \Gamma, \Delta_1, \ldots, \Delta_k$.

Proof: For $k = 0$, we get proposition 2. Let us take for instance for the case of a restricted cut rule. In this case

$$\vdash_{S_1} \Gamma, \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k)$$

follows from

$$\vdash_{S_1} \Gamma', \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k), \mathsf{B}(\sigma)$$

and

$$\vdash_{S_1} \Gamma'', \mathsf{B}'(\sigma_1), \ldots, \mathsf{B}'(\sigma_k), \mathsf{B}'(\sigma)$$

with $\Gamma = \Gamma', \Gamma''$.

By induction hypothesis, we have

$$\vdash_{S_0} \Gamma', \Delta_1, \ldots, \Delta_k, \mathsf{B}(\sigma)$$

and hence by induction hypothesis

$$\vdash_{S_0} \Gamma'', \Gamma', \Delta_1, \ldots, \Delta_k$$

that is

$$\vdash_{S_0} \Gamma, \Delta_1, \ldots, \Delta_k.$$

$\square$

As before, we deduce from this the existence property for (ID).

Conclusion

Why do we need intuitively to extend the system (S_0) to interpret ID? It seems that the intuitionistic notion of ordinals is not enough to represent the closure ordinal of the classical version of B. The definition of ordinals use the notion of function, which is quite different intuitionistically and classically. May be the need of an extension of (S_0) comes from this difference.

Another question is if we really need an inductive definition with "uncountable branching" (from the reference [1, 4], one should expect that it is not needed). This should come from an analysis of the given proof. For instance, it seems that we are really working in the fragment of (S_0) with only positive sequents, and that we never really need to do an induction on a tree with uncountable branching.

It is easier to see this for the Ω-rule: the uncountable branching there comes from the quantification over all proofs in (S_0). If we start from a fixed finitary formulation of a system of inductive definitions, then we can replace this uncountable branching by a countable one.

References

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