A direct proof of Ramsey's Theorem

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Introduction

The infinite version of Ramsey's Theorem is clearly not valid intuitionistically: even in the simple case where we color $\mathbb{N}$ in two colors in a recursive way, one cannot decide which color will appear infinitely often, and even less enumerate an infinite monochromatic subset. However, W. Veldman [3] found an elegant version of Ramsey's Theorem, directly equivalent classically to the infinite version, which is valid intuitionistically. Define a n-ary relation R to be *almost-full* iff for any infinite subset $x_1, x_2, \dots$ we can find $i_1 < \dots < i_k$ such that $R(x_{i_1}, \dots, x_{i_k})$. The intuitionistic Ramsey's Theorem states that the intersection of two almost-full relations is almost full (this can be seen as a generalisation of Dickson's Lemma). This is valid intuitionistically using Brouwer's thesis [3][1] and implies another intuitionistically valid statement of Ramsey's Theorem [1] which can be seen as a generalisation of Paris-Harrigton's Theorem. The goal of this note is to present a simple direct proof of the intuitionistic Ramsey's Theorem. Indeed, this can be seen as a simple proof of the usual version of Ramsey's Theorem.

Intuitionistic Ramsey Theorem

We consider an arbitrary set X, and the set S of finite sequences of elements in X. An element of S is either the empty sequence $()$ or of the form $x\sigma$ for a sequence σ and x element of X. The predicates on S form a distributive lattice for the operation $(A \wedge B)(\sigma) = A(\sigma) \wedge B(\sigma)$ and $(A \vee B)(\sigma) = A(\sigma) \vee B(\sigma)$. If A is a predicate on S and x an element of X we write A_x for the predicate $A_x(\sigma) = A(x\sigma)$ and $A[x]$ for the predicate $A \vee A_x$. To any k-ary relation R on X we associate the predicate $\overline{R}$ on S defined as follows: $\overline{R}(x_1 \dots x_n)$ holds iff $n \geqslant k$ and $R(x_1, \dots, x_k)$ holds. If there is no confusion, we may write simply R for $\overline{R}$.

The set W of well-founded trees over X is defined inductively. The trivial tree $\mathsf{0}$ is in W, and if p_x is a well-founded tree for each x in X, then the tree $\mathsf{sup}(p_x)$ is in W. Given p and q in W we define recursively for each natural number n an element $p \otimes_n q$ of W

- $\mathsf{0} \otimes_0 q = q$ and $(\mathsf{sup}(p_x)) \otimes_0 q = \mathsf{sup}(p_x \otimes q)$
- $\mathsf{0} \otimes_{n+1} q = \mathsf{0}$, $\; p \otimes_{n+1} \mathsf{0} = \mathsf{0}$ and $p \otimes_{n+1} q = \mathsf{sup}((p_x \otimes_{n+1} q) \otimes_n (p \otimes_{n+1} q_x))$ if $p = \mathsf{sup}(p_x)$ and $q = \mathsf{sup}(q_x)$

Using W we can define inductively when a relation is almost-full on X in the following way. We say that p in W *secures* the predicate A on X iff p is $\mathsf{0}$ and $A()$ holds or p is $\mathsf{sup}(p_x)$ and p_x secures $A[x]$ for all x in X [2]. A n-ary relation R is almost-full iff there exists p in W such that p secures $\overline{R}$.

[1] The proof in [3] uses the finite version of Ramsey's Theorem.

[2] Intuitively this means that for any infinite sequence $x_1, x_2, \dots$ we can find $i_1 < \dots < i_k$ such that $A(x_{i_1} \dots x_{i_k})$ holds.

Theorem 0.1 *For any predicate A and B on S, and any n-ary relation R and S on X, if p secures $A \vee R$ and q secures $B \vee S$ then $p \otimes_n q$ secures $A \vee B \vee (R \wedge S)$.*

Proof. The proof is by induction on n and then by induction on p and q. This is direct if $n = 0$. If $n = m + 1$, the difficult case is when $p = \mathsf{sup}(p_x)$ and $q = \mathsf{sup}(q_x)$. In this case p_x secures $A[x] \vee R \vee R_x$ and q_x secures $B[x] \vee S \vee S_x$ for all x. Hence $p_x \otimes_n q$ secures $A[x] \vee B \vee (R \wedge S) \vee R_x$ by induction on p and $p \otimes_n q_x$ secures $A \vee B[x] \vee (R \wedge S) \vee S_x$ by induction on q. By induction on n we have that $(p_x \otimes_n q) \otimes_m (p \otimes_n q_x)$ secures $A[x] \vee B[x] \vee (R \wedge S) \vee (R_x \wedge S_x) = (A \vee B \vee (R \wedge S))[x]$ for all x, hence the result. □

Corollary 0.2 *For any n-ary relation R and S on X, if R and S are almost-full then so if $R \cap S$.*

Comments

Classically, this result implies directly the usual version of Ramsey Theorem. For instance, if we have a 2-coloring $\chi : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ of $\mathbb{N}$, and we define $R_i(n, m)$ to be $\chi(n, m) = i$ then $R_0 \cap R_1$ is the empty relation, so is not almost-full, and so R_0 or R_1 is not almost-full, which gives an infinite monochromatic subset.

This argument is quite similar to the argument proving the so-called clopen version of Ramsey's Theorem in [2] (W. Veldman had independently found an intuitionistic proof of this result). Classically, the clopen version implies the usual infinite Ramsey's Theorem. Intuitionistically, the implication does not seem to hold and this simple argument for Ramsey's Theorem may have some interest.

References

[1] Th. Coquand. An analysis of Ramsey's Theorem. Information and Computation 110(2), p. 297-304, 1994.

[2] Th. Coquand. A Boolean model of ultrafilers. Annals of Pure and Applied Logic, Vol. 99, p. 231-239, 1999.

[3] W. Veldman and M. Bezem. Ramsey's Theorem and the Pigeonhole Principle in Intuitionistic Mathematics. Journal of the London Mathematical Society (2), 47:193-211, 1993.