A direct proof of Ramsey's Theorem

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Introduction

The infinite version of Ramsey's Theorem is clearly not valid intuitionistically: even in the simple case where we color $\mathbb{N}$ in two colors in a recursive way, one cannot decide which color will appear infinitely often, and even less enumerate an infinite monochromatic subset. However, W. Veldman [5] found an elegant version of Ramsey's Theorem, directly equivalent classically to the infinite version, which is valid intuitionistically. Define a n-ary relation R to be *almost-full* iff for any infinite subset $x_1, x_2, \ldots$ we can find $i_1 < \ldots < i_k$ such that $R(x_{i_1}, \ldots, x_{i_k})$. The intuitionistic Ramsey's Theorem states that the intersection of two almost-full relations is almost full (this can be seen as a generalisation of Dickson's Lemma). This is valid intuitionistically using Brouwer's thesis [5][1] and implies another intuitionistically valid statement of Ramsey's Theorem [1] which can be seen as a generalisation of Paris-Harrigton's Theorem. The goal of this note is to present a simple direct proof of the intuitionistic Ramsey's Theorem. Indeed, this can be seen as a simple proof of the usual version of Ramsey's Theorem. Our version generalizes both Veldman's and the clopen version of Ramsey Theorem [2].

Intuitionistic Ramsey Theorem

We consider an arbitrary set X, and the set Σ of finite sequences of elements in X. An element of Σ is either the empty sequence $()$ or of the form $x\sigma$ for a sequence σ and x element of X. The predicates on Σ form a distributive lattice L for the operation $(A \wedge B)(\sigma) = A(\sigma) \wedge B(\sigma)$ and $(A \vee B)(\sigma) = A(\sigma) \vee B(\sigma)$. We write $A \leqslant B$ to mean that $A(\sigma)$ implies $B(\sigma)$ for all σ. If A is a predicate on Σ and x an element of X we write $A \cdot x$ for the predicate $(A \cdot x)(\sigma) = A(x\sigma)$ and $A[x]$ for the predicate $A \vee A \cdot x$. To any k-ary relation R on X we associate the predicate $\overline{R}$ on Σ defined as follows: $\overline{R}(x_1 \ldots x_n)$ holds iff $n \geqslant k$ and $R(x_1, \ldots, x_k)$ holds. If there is no confusion, we may write simply R for $\overline{R}$. We write 1 the predicate on Σ such that $1(\sigma)$ holds for all σ. For two predicates A, B on Σ we have $A = B$ iff $A(\sigma) \leftrightarrow B(\sigma)$ for all σ.

We formulate the results in a more general setting. Let D be a distributive lattice with a top element 1 and an action of X over D, written $a \cdot x$, so that we have

$$(a \vee b) \cdot x = a \cdot x \vee b \cdot x \qquad (a \wedge b) \cdot x = a \cdot x \wedge b \cdot x \qquad 1 \cdot x = 1$$

We define $a[x] = a \vee a \cdot x$. Notice that we have $(a \vee b)[x] = a[x] \vee b[x]$ but we have $(a \wedge b)[x] = (a \wedge b) \vee (a \cdot x \wedge b \cdot x)$ which is not equal to $a[x] \wedge b[x]$ in general.

We define inductively the property $\mathsf{Ar}(a)$ when a has an *arity* (which may be transfinite). We have $\mathsf{Ar}(a)$ iff

1. either $a \cdot x = a$ for all x in X or

[1]The proof in [5] uses the finite version of Ramsey's Theorem.

2. $\mathsf{Ar}(a \cdot x)$ for all x in X.

In the case of the lattice L, it is direct by induction on n that $\mathsf{Ar}(\overline{R})$ for any n-ary relation R. In this case, another way to state the base case, that $A \cdot x = A$ for all x, is that we have $A(\sigma) \leftrightarrow A()$ for any σ.

We define next when the element a is *almost-full*: we have $\mathsf{AF}(a)$ iff

1. either $1 = a$ or

2. $\mathsf{AF}(a[x])$ for all x in X.

Intuitively, for the lattice L, this means that for any infinite sequence $x_1, x_2, \dots$ we can find $i_1 < \dots < i_k$ such that $A(x_{i_1} \dots x_{i_k}\sigma)$ holds for all σ.

Let a, b, r, s be given elements of D.

Lemma 0.1 *If we have $a \leqslant b$ then $\mathsf{AF}(a)$ implies $\mathsf{AF}(b)$.*

Proof. By induction on the proof of $\mathsf{AF}(a)$, using the fact that $a \leqslant b$ implies $a[x] \leqslant b[x]$. $\square$

Lemma 0.2 *If we have $1 = a \vee r$ then $\mathsf{AF}(b \vee s)$ implies $\mathsf{AF}(a \vee b \vee (r \wedge s))$.*

Proof. By induction on the proof of $\mathsf{AF}(b \vee s)$. If we have $1 = b \vee s$ then we have $1 = a \vee b \vee (r \wedge s)$. If we have $\mathsf{AF}(b[x] \vee s[x])$ for all x then we have $\mathsf{AF}(a \vee b[x] \vee (r \wedge s[x]))$ by induction. Since $a \cdot x \vee r \cdot x = 1$ we have $r \wedge s \cdot x \leqslant a \cdot x \vee (r \cdot x \wedge s \cdot x)$ and so

$$a \vee b[x] \vee (r \wedge s[x]) \leqslant (a \vee b \vee (r \wedge s))[x].$$

Usin Lemma 0.1, we get $\mathsf{AF}((a \vee b \vee (r \wedge s))[x])$ for all x, hence the result. $\square$

Lemma 0.3 *If we have $r \cdot x = r$ for all x in X then $\mathsf{AF}(a \vee r)$ and $\mathsf{AF}(b \vee s)$ imply $\mathsf{AF}(a \vee b \vee (r \wedge s))$.*

Proof. By induction on the proof of $\mathsf{AF}(a \vee r)$. Lemma 0.2 deals with the base case where $1 = a \vee r$. If we have $\mathsf{AF}((a \vee r)[x])$ for all x then we have by induction $\mathsf{AF}(a[x] \vee b \vee (r[x] \wedge s))$ and so $\mathsf{AF}(a[x] \vee b \vee (r \wedge s))$ since $r[x] = r$. We have

$$a[x] \vee b \vee (r \wedge s) \leqslant (a \vee b \vee (r \wedge s))[x]$$

and using Lemma 0.1 we get that $\mathsf{AF}((a \vee b \vee (r \wedge s))[x])$ for all x, hence the result. $\square$

Theorem 0.4 *If we have $\mathsf{Ar}(r)$ and $\mathsf{Ar}(s)$ then $\mathsf{AF}(a \vee r)$ and $\mathsf{AF}(b \vee s)$ imply $\mathsf{AF}(a \vee b \vee (r \wedge s))$.*

Proof. By induction first on the proof of $\mathsf{Ar}(r)$ and $\mathsf{Ar}(s)$ and then on the proof of $\mathsf{AF}(a \vee r)$ and $\mathsf{AF}(b \vee s)$. Lemma 0.3 deals with the case where we have $r \cdot x = r$ for all x or $s \cdot x = s$ for all x. Lemma 0.2 deals with the case where $1 = a \vee r$ or $1 = b \vee s$. The remaining case is when $\mathsf{Ar}(r \cdot x)$ and $\mathsf{Ar}(s \cdot x)$ and $\mathsf{AF}(a[x] \vee r[x])$ and $\mathsf{AF}(b[x] \vee s[x])$ for all x. By induction we get $\mathsf{AF}(a[x] \vee b \vee (r[x] \wedge s))$ and $\mathsf{AF}(a \vee b[x] \vee (r \wedge s[x]))$ for all x. Using Lemma 0.1 this implies $\mathsf{AF}(a[x] \vee b[x] \vee (r \wedge s) \vee r \cdot x)$ and $\mathsf{AF}(a[x] \vee b[x] \vee (r \wedge s) \vee s \cdot x)$. By induction we get $\mathsf{AF}(a[x] \vee b[x] \vee (r \wedge s) \vee (r \cdot x \wedge s \cdot x))$ and so $\mathsf{AF}((a \vee b \vee (r \wedge s))[x])$ for all x, hence the result. $\square$

Corollary 0.5 *If we have $\mathsf{Ar}(r)$ and $\mathsf{Ar}(s)$ then $\mathsf{AF}(r)$ and $\mathsf{AF}(s)$ implies $\mathsf{AF}(r \wedge s)$.*

Comments

Classically, this result implies directly the usual version of Ramsey Theorem. For instance, if we have a 2-coloring $\chi : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ of $\mathbb{N}$, and we define $R_i(n, m)$ to be $n = m \vee \chi(n, m) = i$ then $R_0 \cap R_1$ is the empty relation, so it is not almost-full, and so R_0 or R_1 is not almost-full, which gives an infinite monochromatic subset.

This argument is quite similar to the argument proving the so-called clopen version of Ramsey's Theorem in [2] (W. Veldman had independently found an intuitionistic proof of this result), and is also similar to the argument presented in [6] (but without explicit mention to choice sequences). Classically, the clopen version implies the usual infinite Ramsey's Theorem. Intuitionistically, this implication does not seem to hold and the argument presented here is a common generalization of both the clopen version and Veldman's intuitionistic Ramsey Theorem.

We end by the following conjecture, inspired by Hindman's Theorem [3], as generalized by Milliken [4]. If X is furthermore a commutative monoid, and we have another action $r + x$ on D, satisfying

$$(r + x) + y = r + (x + y) \qquad (r + x) \cdot y = r \cdot (x + y)$$

and if we redefine $\mathsf{AF}(r)$ as either $1 = r$ or $\mathsf{AF}(r \vee (r \cdot x) \vee (r + x))$ for all x, we still have then that the elements satisfying both Ar and AF are closed under the meet operation.

References

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[3] N. Hindman. Finite sums from sequences within cells of a partition of N. Journal of Combinatorial Theory (A), vol. 17, p. 1-11, 1974.

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[6] W. Veldman. An intuitionistic proof of Kruskal's theorem. Arch. Math. Logic 43, 215-264 (2004).