On the Decidability of Conversion in Type Theory

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We present a proof of decidability of conversion in dependent type theory with natural numbers N, a universe U à la Russell, and functional extensionality. Our approach is based on one by Abel and Scherer [1] with the difference that we use *typed* weak head reduction and thus get subject reduction for free. First we define one logical relation and prove it sound and complete w.r.t. the type system. We then obtain canonicity, injectivity of Π, and unicity of types for neutral terms. We then define algorithmic equality and show it is decidable. We define a second logical relation and show it is sound and complete w.r.t. the type system and algorithmic equality. We thus conclude that conversion is decidable in the type system.

Typed weak head reduction $\Gamma \vdash t \to t' : A$, which is a subrelation of conversion $\Gamma \vdash t = t' : A$, is given by:

$$\frac{\Gamma \vdash t \to t' : \Pi(x{:}A)B \qquad \Gamma \vdash a : A}{\Gamma \vdash t\,a \to t'a : B[a]} \qquad \frac{\Gamma, x{:}A \vdash t : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x.t)\,a \to t[a/x] : B[a]}$$

$$\frac{\Gamma \vdash t \to u : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t \to u : B} \qquad \frac{\Gamma \vdash A \to B : \mathsf{U}}{\Gamma \vdash A \to B}$$

We also define the usual reflexive-transitive closures $\to^*$ of these relations.

We define a logical relation by mutual induction-recursion. Inductively we define $\Gamma \Vdash A$ by introduction rules. By recursion on the proofs of $\Gamma \Vdash A$ we define $\Gamma \Vdash a : A$, $\Gamma \Vdash a = b : A$ and $\Gamma \Vdash A = B$.

Definition 1. $$\textsc{f-N}\ \frac{\Gamma \vdash A \to^* \mathsf{N}}{\Gamma \Vdash A} \qquad \textsc{f-Neu}\ \frac{\Gamma \vdash A \to^* K \qquad K \text{ neutral}}{\Gamma \Vdash A} \qquad \textsc{f-U}\ \frac{\Gamma \vdash A \to^* \mathsf{U}}{\Gamma \Vdash A}$$

$$\textsc{f-}\Pi\ \frac{\begin{array}{l}\Gamma \vdash A \to^* \Pi(x{:}F)G \ \ \Gamma \Vdash F \\ (\forall a, \forall \Delta \leq \Gamma)(\Delta \Vdash a : F \Rightarrow \Delta \Vdash G[a]) \ \ (\forall a, b, \forall \Delta \leq \Gamma)(\Delta \Vdash a = b : F \Rightarrow \Delta \Vdash G[a] = G[b])\end{array}}{\Gamma \Vdash A}$$

- If $\Gamma \Vdash A$ by f-Neu ($A \to^* K$ with K neutral) then
  - $\Gamma \Vdash A = B$ if $\Gamma \vdash B \to^* L$ with L neutral and $\Gamma \vdash K = L$.
  - $\Gamma \Vdash t : A$ if $\Gamma \vdash t \to^* l : A$ with l neutral.
  - $\Gamma \Vdash t = u : A$ if $\Gamma \vdash t \to^* l : A$ and $\Gamma \vdash u \to^* k : A$ with l and k neutrals and $\Gamma \vdash l = k : A$.
- If $\Gamma \Vdash A$ by f-N then
  - $\Gamma \Vdash A = B$ if $\Gamma \vdash B \to^* \mathsf{N}$.
  - $\Gamma \Vdash t : A$ if one of (i.) $\Gamma \vdash t \to^* 0 : A$ (ii.) $\Gamma \vdash t \to^* \mathsf{S}\,u : A$ and $\Gamma \Vdash u : A$ (iii.) $\Gamma \vdash t \to^* k : A$ with k neutral.
  - $\Gamma \Vdash t = u : A$ if one of (i.) $\Gamma \vdash t \to^* 0 : A$ and $\Gamma \vdash u \to^* 0 : A$ (ii.) $\Gamma \vdash t \to^* \mathsf{S}\,t' : A$, $\Gamma \vdash u \to^* \mathsf{S}\,u' : A$ and $\Gamma \Vdash t' = u' : A$ (iii.) $\Gamma \vdash t \to^* k : A$ and $\Gamma \vdash u \to^* l : A$ with l and k neutral and $\Gamma \vdash k = l : A$.

- If $\Gamma \Vdash A$ by F-Π with $\Gamma \vdash A \to^* \Pi(x{:}F)G$ then
  - $\Gamma \Vdash A = B$ if $\Gamma \Vdash B$ with $\Gamma \vdash B \to^* \Pi(x{:}H)E$ and $\Gamma \Vdash F = H$ and $\Delta \Vdash G[a] = E[a]$ whenever $\Delta \Vdash a : F$ for any a and $\Delta \leq \Gamma$.
  - $\Gamma \Vdash f : A$ if $\Gamma \vdash f : A$ and $\Delta \vdash f\, a : G[a]$ whenever $\Delta \Vdash a : F$, and $\Delta \vdash f\, a = f\, b : G[a]$ whenever $\Delta \Vdash a = b : F$ for any a, b and $\Delta \leq \Gamma$.
  - $\Gamma \Vdash f = g : A$ if $\Gamma \Vdash f : A$ and $\Gamma \Vdash g : A$ and $\Delta \Vdash f\, a = g\, a : G[a]$ whenever $\Delta \Vdash a : F$ for any a and $\Delta \leq \Gamma$.

Formally, $\Gamma \Vdash A$ is the type of a derivation. E.g., the rule F-N should be written as $\dfrac{\mathcal{R} :: \Gamma \vdash A \to^* \mathsf{N}}{\mathrm{tf}_{\Gamma,A}(\mathcal{R}) :: \Gamma \Vdash A}$ where $\mathcal{R}$ is a proof of $\Gamma \vdash A \to^* \mathsf{N}$. The forcing $\Gamma \Vdash t : A$ is then given by a map $\mathrm{trm} : (\Gamma \in \mathbb{C},\ A \in \mathbb{T},\ \mathcal{F} \in (\Gamma \Vdash A)) \to \mathcal{P}(\mathbb{T}))$ defined by recursion on $\mathcal{F}$, where $\mathbb{C}$ and $\mathbb{T}$ are the sets of contexts and terms and $\mathcal{P}$ denotes the powerset operation. A proof irrelevance result showing $\mathrm{trm}(\Gamma, A, \mathcal{F}) = \mathrm{trm}(\Gamma, A, \mathcal{F}')$ for any two derivations $\mathcal{F}$ and $\mathcal{F}'$ of $\Gamma \Vdash A$ then allows us to write $\Gamma \Vdash t : A$ whenever $t \in \mathrm{trm}(\Gamma, A, \mathcal{F})$.

Since subject reduction is immediate, soundness is also immediate from the definition. Completeness follows from the usual fundamental theorem of logical relations (long proof).

Lemma 2 (Soundness)**.** *If* $\Gamma \Vdash J$ *then* $\Gamma \vdash J$*.*

Theorem 3 (Completeness)**.** *If* $\Gamma \vdash J$ *then* $\Gamma \Vdash J$*.*

Corollary 4 (Canonicity)**.** *If* $\vdash t : \mathsf{N}$ *then* $\vdash t \to^* \mathsf{S}^k\, 0 : \mathsf{N}$ *for some* k*.*

Corollary 5 (The function type constructor Π is injective)**.** *If* $\Gamma \vdash \Pi(x{:}F)G = \Pi(x{:}F')G'$ *then* $\Gamma \vdash F = F'$ *and* $\Gamma, x{:}F \vdash G = G'$*.*

We then define algorithmic equality $\Gamma \vdash A \;\mathsf{conv}\; B$ and $\Gamma \vdash a \;\mathsf{conv}\; b : A$. Intuitively it says that two terms are convertible if the have a common normal form. Soundness and conversion have simple inductive proofs, thanks to the meta theory established with the logical relation.

Lemma 6 (Soundness of algorithmic equality)**.** *If* $\Gamma \vdash A \;\mathsf{conv}\; B$ *then* $\Gamma \vdash A = B$ *and if* $\Gamma \vdash a \;\mathsf{conv}\; b : A$ *then* $\Gamma \vdash a = b : A$*.*

Lemma 7 (Conversion is decidable)**.** *If* $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *then* $\Gamma \vdash A \;\mathsf{conv}\; B$ *is decidable. If* $\Gamma \vdash a : A$ *and* $\Gamma \vdash b : A$ *then* $\Gamma \vdash a \;\mathsf{conv}\; b : A$ *is decidable.*

On top of algorithmic equality we define a second logical relation similarly to Definition 1. The one major difference is that for two neutral terms k and l of some base type, say N, to satisfy $\Gamma \Vdash k = l : \mathsf{N}$ they need not only to be judgmentally equal but also convertible. After proving the fundamental theorem for this second logical relation, we get completeness of algorithmic equality.

Lemma 8. *If* $\Gamma \vdash A = B$ *then* $\Gamma \vdash A \;\mathsf{conv}\; B$ *and if* $\Gamma \vdash a = b : A$ *then* $\Gamma \vdash a \;\mathsf{conv}\; b : A$*.*

Theorem 9. *In type theory, judgemental equality is decidable.*

References

[1] A. Abel and G. Scherer. On irrelevance and algorithmic equality in predicative type theory. *Logical Methods in Computer Science*, 8(1):1–36, 2012. TYPES'10 special issue.